

FLOW THROUGH A CIRCULAR PIPE

Hagen Poisselle’s theory is based on the following assumptions :

- (1) The fluid flow follows Newton’s Law of viscosity.
 Newton’s law of viscosity states that , the shear stress τ *in fluid* element is directly proportional to the rate of shear strain i.e $\tau = \mu \frac{\partial u}{\partial y}$
- (2) There is no slip of fluid particles at the boundary (fluid particles adjacent to the pipe wall have zero viscosity).

Let us consider a horizontal pipe as shown in figure 2.

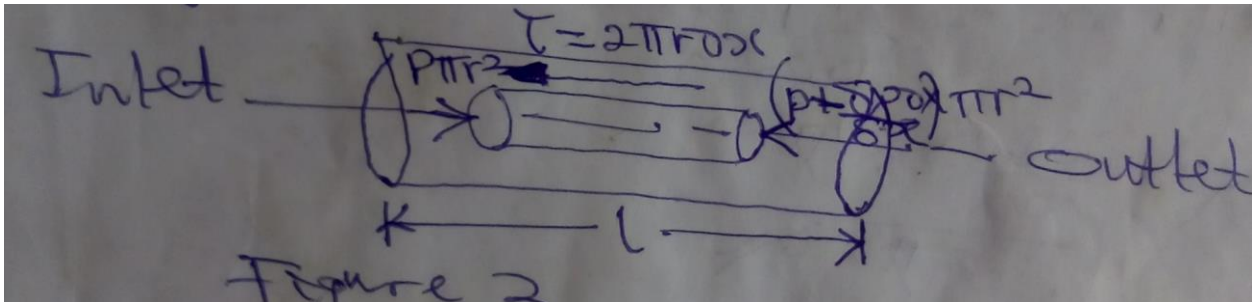


Figure 2

Figure 2 shows a horizontal circular pipe of radius ‘R’ having laminar flow of fluid through it. Also consider a small concentric cylinder (fluid element) of radius “r” and distance “x” as a free body shear force “F” is given as $F = \tau \cdot 2\pi r \cdot \delta x$

Let P be the intensity of the pressure at the left end and $P + \frac{\partial P}{\partial x}$ be the pressure at the right hand side end.

Forces acting on the fluid element are as follows :

- (1) The shear force $F = \tau \cdot 2\pi r \cdot \delta x$ on the surface of fluid element .
- (2) Pressure force $P \pi r^2$ on the left end .
- (3) Pressure force $(P + \frac{\partial P}{\partial x} \delta x) \pi r^2$ on the right end.

For a steady flow, the net forces on the cylinder must be equal to zero.

$$P \pi r^2 - \tau \cdot 2\pi r \cdot \delta x - (P + \frac{\partial P}{\partial x} \delta x) \pi r^2 = 0 \quad \text{----- (1)}$$

$$P \pi r^2 - \tau \cdot 2\pi r \cdot \delta x - P \pi r^2 - \frac{\partial P}{\partial x} \delta x \pi r^2 = 0 \quad \text{----- (2)}$$

$$-\tau \cdot 2\pi r \cdot \delta x = \frac{\partial P}{\partial x} \delta x \pi r^2 \quad \text{----- (3a)}$$

$$-\tau = \frac{\frac{\partial P}{\partial x} \delta x \pi r^2}{2\pi r \cdot \delta x}$$

$$-\tau = \frac{\partial P r}{\partial x 2}$$

$$\tau = -\frac{\partial P r}{\partial x 2} \text{ ----- (3b)}$$

Recall from the Newton's law of viscosity

$$\tau = \mu \frac{\partial u}{\partial y} \text{ ----- (4)}$$

$$y = R - r \text{ -----(5a)}$$

When radius R = 0

$$y = -r \text{ ----- (5b)}$$

$$\delta y = -\delta r \text{ ----- (5c)}$$

Substitute equation (5c) into (4)

$$\tau = -\mu \frac{\partial u}{\partial r} \text{ -----(6)}$$

Comparing (3b) and (6)

$$-\mu \frac{\partial u}{\partial r} = -\frac{\partial P r}{\partial x 2}$$

$$\mu \delta u = \frac{\partial P r}{\partial x 2} \delta r \text{ ----- (7)}$$

$$\delta u = \frac{\partial P r}{\partial x 2\mu} \delta r \text{ ----- (8)}$$

Integrate with respect to r

$$u = \frac{1}{2\mu} \frac{\partial P}{\partial x} \frac{r^2}{2} + C$$

$$u = \frac{1}{4\mu} \frac{\partial P}{\partial x} r^2 + C \text{ ----- (9)}$$

At the boundary condition r = R and u = 0

$$0 = \frac{1}{4\mu} \frac{\partial P}{\partial x} R^2 + C$$

$$C = -\frac{1}{4\mu} \frac{\partial P}{\partial x} R^2 \text{ ----- (10)}$$

Substitute equation (10) and (9)

$$u = \frac{1}{4\mu} \frac{\partial P}{\partial x} r^2 - \frac{1}{4\mu} \frac{\partial P}{\partial x} R^2 \text{ ----- (11)}$$

$$u = \frac{1}{4\mu} \frac{\partial P}{\partial x} (r^2 - R^2) \text{ ----- (12a)}$$

$$u = -\frac{1}{4\mu} \frac{\partial P}{\partial x} (R^2 - r^2) \text{ ----- (12b)}$$

For maximum velocity ; $r = 0$

$$u_{max} = -\frac{1}{4\mu} \frac{\partial P}{\partial x} R^2 \text{ ----- (12c)}$$

Equation (12b) and that of (12c) are fore velocity and maximum velocity for the fluid in a circular pipe respectively .

Divide equation (12b) by (12c)

$$\frac{u}{u_{max}} = \frac{-\frac{1}{4\mu} \frac{\partial P}{\partial x} (R^2 - r^2)}{-\frac{1}{4\mu} \frac{\partial P}{\partial x} R^2}$$

$$\frac{u}{u_{max}} = \frac{(R^2 - r^2)}{R^2}$$

$$\frac{u}{u_{max}} = 1 - \frac{r^2}{R^2}$$

$$u = u_{max} \left(1 - \frac{r^2}{R^2} \right) \text{ ----- (13)}$$

LAMINAR FLOW IN CIRCULAR PIPE

For discharge Q

The discharge through an elementary ring of thickness " δr " at radial discharge " r " is given by :

$$\delta Q = 2\pi r u \delta r \text{ -----(14)}$$

Substitute (13) into (14)

$$\delta Q = u_{max} \left(1 - \frac{r^2}{R^2} \right) 2\pi r \delta r \text{ ----- (15a)}$$

$$\int dQ = \int_0^R u_{max} \left(1 - \frac{r^2}{R^2} \right) 2\pi r \delta r \text{ ----- (15b)}$$

$$Q = u_{max} 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R \text{ ----- (16)}$$

$$Q = u_{max} 2\pi \left[\frac{R^2}{2} - \frac{R^4}{4R^2} \right]$$

$$Q = 2\pi u_{max} \left[\frac{2R^4 - R^4}{4R^2} \right]$$

$$Q = 2\pi u_{max} \left[\frac{R^4}{4R^2} \right]$$

$$Q = 2\pi u_{max} \left[\frac{R^2}{4} \right]$$

$$Q = \frac{\pi}{2} u_{max} R^2 \text{ ----- (17)}$$

THE AVERAGE VELOCITY OF FLOW

$$\bar{u} = \frac{Q}{A} \quad (\text{continuity equation})$$

$$\bar{u} = \frac{\frac{\pi}{2} u_{max} R^2}{\pi R^2}$$

$$\bar{u} = \frac{u_{max}}{2} \quad \text{----- (18) For a circular pipe}$$

By substituting equation (12c) into equation (18)

$$\bar{u} = \frac{-\frac{1}{4\mu} \frac{\partial P}{\partial x} R^2}{2}$$

$$\bar{u} = -\frac{1}{8\mu} \frac{\partial P}{\partial x} R^2 \quad \text{----- (19)}$$

$$-\delta P = \frac{8\mu \bar{u} \delta x}{R^2} \quad \text{----- (20)}$$

The pressure difference between 2 sections 1 and 2 at a distance x1 and x2 is given as :

$$-\int_{P1}^{P2} \delta P = \int_{X1}^{X2} \frac{8\mu \bar{u} \delta x}{R^2}$$

$$P1 - P2 = \frac{8\mu \bar{u}}{R^2} [X2 - X1] \quad \text{----- (21)}$$

Note : X2 - X1 = L and P1 - P2 = ΔP Hence :

$$\Delta P = \frac{8\mu \bar{u} L}{R^2} \quad \text{----- (22)}$$

Also note that ; $\frac{D}{2} = R$

$$\Delta P = \frac{8\mu \bar{u} L}{\left[\frac{D}{2}\right]^2}$$

$$\Delta P = \frac{32\mu \bar{u} L}{D^2} \quad \text{----- (23)}$$

$$\bar{u} = \frac{Q}{A}$$

$$\Delta P = \frac{32\mu \frac{Q}{A} L}{D^2}$$

$$\Delta P = \frac{32\mu \bar{Q} L}{D^2 A} \quad \text{----- (24)}$$

$$A = \frac{\pi D^2}{4}$$

$$\Delta P = \frac{32\mu \bar{Q} L}{D^2 \cdot \frac{\pi D^2}{4}}$$

$$\Delta P = \frac{128 \mu \bar{Q} L}{\pi D^4} \quad \text{----- (25)}$$

Equation (23) and (24) are referred to as Hagen Poiseuille's equation .

Example 1

Oil of viscosity 700 centipoise and specific gravity 0.9 flow through a horizontal pipe of diameter 80 mm. If the pressure drop in 100m length of pipe is 1800 kN/m² . Calculate :

- (a) Rate of flow of oil
- (b) Centre line velocity (maximum velocity)
- (c) Total frictional drag over 100m length
- (d) Power required to maintain the flow
- (e) The velocity gradient at the pipe wall
- (f) The velocity and shear stress at 80 mm from the wall

SOLUTION

Coefficient of dynamic viscosity " μ " = 700 cp

$$1000\text{cp} = 1 \text{ Ns/m}^2$$

$$700 \text{ cp} = \mu$$

$$\mu = \frac{700}{1000} = 0.7 \text{ Ns/m}^2$$

$$\frac{\partial P}{\partial x} = \frac{1800 \times 10^3}{100}$$

$$\frac{\partial P}{\partial x} = -\frac{1800 \times 10^3}{100}$$

Pipe diameter $D = 80 \text{ mm} = 0.08 \text{ m}$

Specific gravity of oil = 0.9

Density of the oil $\rho = 0.9 \times 10^3 \text{ kg/m}^3$

$$\text{Area of the pipe } A = \frac{\pi D^2}{4}$$

$$A = \frac{3.142 \times 0.08^2}{4} = 0.0050 \text{ m}^2$$

(a) Rate of flow of oil = $Q = A \bar{u}$

$$\bar{u} = -\frac{1}{8\mu} \frac{\partial P}{\partial x} R^2$$

$$\bar{u} = \left(-\frac{1}{8 \times 0.07}\right) \left(-\frac{1800 \times 10^3}{100}\right) (0.04^2)$$

$$\bar{u} = 5.14 \text{ m/s}$$

$$Q = 0.0050 \times 5.14$$

$$Q = 0.0257 \text{ m}^3/\text{s}$$

Alternatively

$$\Delta P = \frac{32\mu \bar{u} L}{D^2}$$

$$1800 = \frac{32 \times 0.7 \mu \bar{u} \times 100}{0.08^2}$$

$$\bar{u} = 5.14 \text{ m/s}$$

$$Q = A\bar{u}$$

$$Q = 0.0050 \times 5.14$$

$$= 0.0257 \text{ m}^3/\text{s}$$

Nature of flow

$$Re = \frac{\rho u D}{\mu}$$

$$Re = \frac{0.9 \times 10^3 \times 5.14 \times 0.08}{0.7}$$

$$Re = 528.69$$

Since $Re < 2000$

The flow is laminar

(b) Centre line velocity

$$u_{max} = 2 \bar{u}$$

$$= 2 \times 5.14$$

$$u_{max} = 10.28 \text{ m/s}$$

(c) Total frictional drag F_D

$$F_D = \tau_o \pi D L$$

$$\tau_o = - \frac{\partial P r}{\partial x 2}$$

$$\tau_o = \frac{1800 \times 10^3 \times 0.04}{100 \times 2}$$

$$\tau_o = 360 \text{ N/m}^2$$

$$F_D = 360 \times 3.142 \times 0.08 \times 100$$

$$F_D = 9047.7 \text{ N}$$

$$F_D \approx 9.048 \text{ kN}$$

(d) Power required to maintain the flow

$$P = F_D \cdot \bar{u}$$

$$P = 9047.7 \times 5.14$$

$$P = 46505 \text{ Watts}$$

$$P = 46.5 \text{ kW}$$

(e) Velocity gradient at the pipe wall

$$\tau_o = \mu \frac{\partial u}{\partial y} \quad @ \ y = 0$$

$$\frac{\partial u}{\partial y} = \frac{\tau_o}{\mu}$$

$$\frac{\partial u}{\partial y} = \frac{360}{0.7}$$

$$\frac{\partial u}{\partial y} = 514.2857 \text{ s}^{-1}$$

(f) Velocity at 80 mm from the wall

$$u = -\frac{1}{4\mu} \frac{\partial P}{\partial x} (R^2 - r^2)$$

$$y = 0.08 \text{ m}$$

$$y = R - r$$

$$0.08 = 0.04 - r$$

$$-0.04 = r$$

$$u = -\frac{1}{4 \times 0.7} \left(-\frac{1800 \times 10^3}{100} \right) (0.04^2 - (-0.04)^2)$$

$$u = 0 \text{ m/s}$$

(g) Shear stress at 80 mm from the wall

$$\frac{\tau_{mm}}{r} = \frac{\tau_o}{R}$$

$$\frac{\tau_{mm}}{0.04} = \frac{360}{0.04}$$

$$\tau_{mm} = 360 \text{ N/m}^2$$

