(2)

Integration by Part

For integrating product of simple functions such as $\int xe^x dx$, $\int t\sin t\,dt$, $\int e^\theta \cos\theta\, d\theta$ etc, we use the integration by part formular which is given as

$$\int u\,dv = uv - \int v\,du$$

Guidelines for selecting u and dv (There are exceptions but this is helpful)

LIATE (choose u to be the fn that comes first in this list)

L - Logarithm function e.g $\log_e x$ $(\ln x)$

I - Inverse trig functions e.g $\sin^{-1}x$, $\cos^{-1}x$, etc

A - Algebraic functions e.g x, x^2, x^3, x^4 etc

T - Trigonometric fns e.g $\sin x$, $\cos x$, etc

E - Exponential functions e.g e^{2x}, e^{3x}, etc

Example 1

$$\int x\cos x\,dx$$

$u = x$, $dv = \cos x$

$\frac{du}{dx} = 1$ $\quad v = \sin x$

$$\int u\,dv = uv - \int v\,du$$

$$= x(\sin x) - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

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$$\int x^2 \log_e x\,dx$$

$u = \log_e x$ $\quad dv = x^2$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^3}{3}$

$$\int u\,dv = uv - \int v\,du$$

$$= \log_e x\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

we have

$$\frac{x^3}{3}\log_e x - \int \frac{x^2}{3}dx$$

$$= \frac{x^3}{3}\log_e x - \frac{x^3}{9} + C$$

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$$\int x^3 e^x dx$$

$u = x^3$, $dv = e^x$

$du = 3x^2 dx$ $\quad v = e^x$

$$\int u\,dv = uv - \int v\,du$$

$$= x^3(e^x) - \int e^x \cdot 3x^2 dx$$

$$= x^3e^x - \int 3x^2e^x dx$$

$x^3e^x -$ $\begin{cases} u = 3x^2, & dv = e^x \\ du = 6x\,dx & v = e^x \\ \Rightarrow 3x^2(e^x) - \int e^x \cdot 6x\,dx \end{cases}$

$$x^3e^x - 3x^2e^x + \int 6xe^x dx$$

$\begin{cases} u = 6x & dv = e^x \\ du = 6dx & v = e^x \\ 6xe^x - \int e^x \cdot 6dx \end{cases}$

$$x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

(4)

Evaluate $\int e^x \sin x\,dx$

$u = \sin x$, $dv = e^x$

$du = \cos x\,dx$, $v = e^x$

$\int u\,dv = uv - \int v\,du$

$\sin x\,(e^x) - \int e^x \cos x\,dx$

$e^x \sin x - \int e^x \cos x\,dx$

$u = \cos x \quad dv = e^x$

$du = -\sin x\,dx \quad v = e^x$

$\cos x\,(e^x) - \int e^x(-\sin x)\,dx$

$e^x \cos x + \int e^x \sin x\,dx$

$e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

Let $I = \int e^x \sin x\,dx$

$I = e^x \sin x - e^x \cos x - I$

$2I = e^x \sin x - e^x \cos x$

$$I = \frac{e^x \sin x - e^x \cos x}{2}$$

Thus

$$\int e^x \sin x\,dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x\right] + c$$

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a) $\int 2x^2 \ln x\,dx$ $\quad \left[\frac{2}{3}x^3\left(\ln x - \frac{1}{3}\right)\right] + c$

b) $\int 3te^{2t}\,dt$ $\quad \left[\frac{3}{2}te^{2t} - \frac{3e^{2t}}{4}\right] + c$

c) $\int x^2 \sin x\,dx$

Integration of Trig functions

If the integrand is a product of sine or cosine of multiple angles, it may be expressed as a sum by means of the Identities

1. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
2. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
4. $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

Examples

(1) Evaluate $\int \sin 5x \sin x\,dx$

$A = 5x$, $B = 1x$

$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

$= -\frac{1}{2}[\cos 6x - \cos 4x]$

$\int \sin 5x \sin x\,dx = -\frac{1}{2}\int(\cos 6x - \cos 4x)\,dx$

$= -\frac{1}{2}\left[\frac{\sin 6x}{6} - \frac{\sin 4x}{4}\right]$

$= \frac{-\sin 6x}{12} + \frac{\sin 4x}{8} + c$

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$\int \sin 3x \cos x\,dx$

$= \frac{-\cos 4x}{8} - \frac{\cos 2x}{4} + c$

(3)

a) $\int \cos 5x \cos 6x\,dx$

b) $\int \sin 7x \cos 2x\,dx$