RESONANCE

A notable feature of the frequency response of a circuit is the sharp (resonant) peak exhibited in the amplitude characteristics. Resonance occurs in any system that has a complex conjugate pair of poles. i.e. any circuit with at least one inductor and one capacitor.

For an RLC circuit, resonance occurs when the capacitive and inductive reactances are equal in magnitude, resulting in a purely resistive circuit.

Series or parallel resonant circuits are useful for filter construction due to their highly frequency-selective transfer function characteristics.

Series Resonance

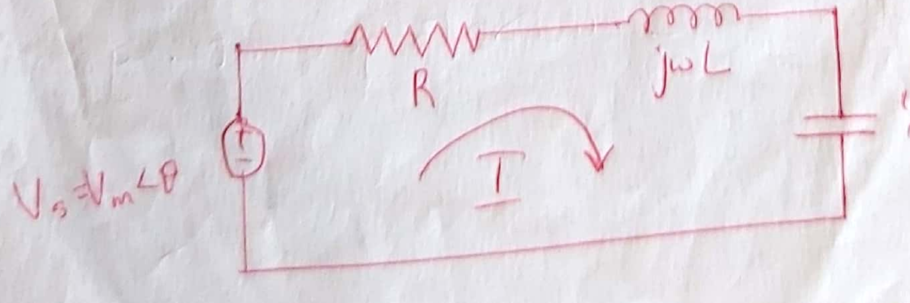

Fig: Series Resonant Circuit

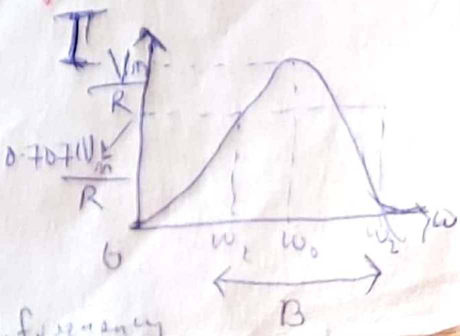

Fig: Current Amplitude v. frequency

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i.e. $\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2} = \sqrt{2}\cdot R$

for the circuit,

$$H(\omega) = Z = \frac{V_s}{I} = R + j\omega L + \frac{1}{j\omega C} \quad \text{—— (i)}$$

$$= R + j\left(\omega L + \frac{1}{\omega \cdot j^2}\right)$$

$$= R + j\left(\omega L - \frac{1}{\omega C}\right)$$

At resonance, the imaginary part of $H(\omega) = 0$

$$\Rightarrow \quad Im(H(\omega)) \equiv I_m(Z) = \omega L - \frac{1}{\omega C} = 0$$

$$\Rightarrow \quad \omega_0 L = \frac{1}{\omega_0 C} \quad \text{—— (ii)}$$

ω_0 = resonant frequency is the value of ω that satisfies this condition:

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s} \quad \text{—— (iii)}$$

but $\omega_0 = 2\pi f_0$

$$\Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ Hz} \quad \text{—— (iv)}$$

At resonance,

- The LC series combination becomes a short circuit, and the entire voltage is across R.

- Power factor becomes unity because voltage, V_s and current, I are in phase.

- The transfer function has a minimum magnitude value $H(\omega) = Z(\omega) = R$

The current, $I = |I| = \dfrac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$ — (i)

The average power dissipated by the RLC circuit is:

$$P(\omega) = \frac{1}{2} I^2 R \quad \text{— (ii)}$$

The highest (maximum) power is dissipated at resonance, when

$I = V_m/R$

$\Rightarrow P(\omega_0) = \dfrac{1}{2} \dfrac{V_m^2}{R}$ — (iii) $\left[\text{i.e } \left(\dfrac{V_m}{\sqrt{2}}\right)^2 / R\right]$

$\Rightarrow P(\omega_0) = \dfrac{1}{2} \dfrac{V_m}{R} \cdot \dfrac{V_m}{R} \cdot R = \dfrac{1}{2} V_m^2 / R$

At frequencies $\omega = \omega_1, \omega_2$, the dissipated power becomes half the maximum value, i.e.

$$P(\omega_1) = P(\omega_2) = \left[\frac{(V_m/\sqrt{2})^2}{2R}\right] = \frac{V_m^2}{4R} \quad \text{— (iv)}$$

$$\frac{V_m^2}{2 \cdot 2 \cdot R} = \frac{P(\omega_0)}{2}$$

$\therefore$ ω_1 & ω_2 are referred to as half-power frequencies and are obtained by setting $Z = R \cdot \sqrt{2}$; $\Rightarrow I_{\omega_{1,2}} = \dfrac{V_m}{\sqrt{2} \cdot R}$

hence, half-power current value is : $I_{\omega_1,\omega_2} = \frac{V_m}{R \cdot \sqrt{2}}$

ω_1 and ω_2 are given by:

$$\left.\begin{aligned} \omega_1 &= \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \; ; \\ \omega_2 &= \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \end{aligned}\right\} \text{—— (ix)}$$

and, $\omega_0 = \sqrt{\omega_1 \cdot \omega_2}$ —— (x)

i.e. the resonant frequency is the geometric mean of the half-power frequencies.

ω_1, ω_2 are approximately symmetrical around the resonant frequency, ω_0.

Also,

$$B = \omega_2 - \omega_1 \quad \text{—— (xi)}$$

B = bandwidth i.e. half-power bandwidth.

The quality factor, Q is a quantitative measure of the "sharpness" of the resonance in a resonant circuit. At resonance, the reactive energy oscillates between the inductor and the capacitor.

Quality factor, Q is given by:

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}}$$

It is a measure of the energy storage capacity of a circuit compared to the energy dissipation property.

hence, $Q = 2\pi \dfrac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R(1/f_0)} = \dfrac{2\pi f_0 L}{R}$ —— (xii)

$\Rightarrow Q = \dfrac{W_0 L}{R} = \dfrac{1}{W_0 R C}$ $\left(W_0 L = \dfrac{1}{W_0 C}\right)$ —— (xiii)

Q is dimensionless.

where $B = W_2 - W_1 = \dfrac{R}{L}$ —— (xiv) (from eq. ix)

$\Rightarrow B = \dfrac{R}{L} = \dfrac{W_0}{Q}$ —— (xv)

or $B = W_0^2 CR$ —— (xvi)

and $Q = \dfrac{W_0}{B}$

& (xvi)

NB: Eqs. (ix), (xiii) and (xv) only apply to series RLC circuits.

Resonant circuits are designed to operate at or near their ~~the~~ resonant frequencies. High values of Q implies high selectivity and smaller bandwidth, and vice-versa.

Thus, the selectivity of an RLC circuit is the ability of the circuit to respond to a certain frequency, and discriminate against all other frequencies.

For ~~the~~ circuits with high Q i.e. $Q \geq 10$, the half-power frequencies, for all practical intents, are symmetrical around the resonant frequency. They are approximated as:

$$\omega_1 \simeq \omega_0 - \frac{B}{2} \quad ; \quad \omega_2 = \omega_0 + \frac{B}{2} \qquad \text{—(xviii)}$$

Circuits with high Q are useful in communication networks to accurately & precisely select the desired frequency of Xmission

Thus, five parameters — ω_0, ω_1, ω_2, B and Q characterise resonant circuits.

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Examples:

A series RLC circuit has $R = 2\Omega$, $L = 1mH$ & $C = 0.4\mu F$.
Determine:

(i) the resonant frequency and the half-power frequencies

(ii) the quality factor and bandwidth

(iii) The amplitude of the current at ω_0, ω_1 & ω_2.
(highest/maximum current) (half-power current)

Take $V = 20 \sin \omega t$.

Solution

(i) resonant frequency, ω_0:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = \underline{\underline{50\,krad/s}}$$

half-power frequencies, ω_1 & ω_2:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

and

$$\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = \sqrt{\left(\frac{2}{2 \cdot 10^{-3}}\right)^2 + \frac{1}{10^{-3} \cdot 0.4 \times 10^{-6}}}$$

$$= \sqrt{(10^3)^2 + (50 \times 10^3)^2} = 10^3 \sqrt{1 + 2500}$$

$$\Rightarrow \omega_1 = \frac{-2}{2 \times 10^{-3}} + 10^3 \sqrt{1 + 2500}$$

$$= -1 \times 10^3 + 10^3 \sqrt{1 + 2500}$$

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$$\Rightarrow \omega_1 = \left(-1 + \sqrt{1+2500}\right) \times 10^3 \text{ rad/s}$$

$$\simeq -49 \text{ Krad/s}$$

Similarly,

$$\omega_2 = 1 + \sqrt{1+2500} \text{ krad/s}$$

$$\simeq 51 \text{ Krad/s}$$

(ii) Bandwidth:

$$B = \omega_2 - \omega_1 = (51 - 49) \text{ Krad/s}$$

$$= 2 \text{ Krad/s}$$

OR

$$B = R/L = 2/10^{-3} = 2\text{K rad/s}$$

$$Q = \omega_0/B = \frac{50 \text{ Krad/s}}{2 \text{ Krad/s}} = 25$$

(iii) At $\omega = \omega_0$, $(V_m = 20V)$ $(V = V_m \sin \omega t)$

$$I = V_m/R = 20/2$$

$$= 10A$$

At $\omega = \omega_1, \omega_2$,

$$I = \frac{V_m}{\sqrt{2}R} = \frac{20}{2\sqrt{2}} = 10/\sqrt{2}$$

$$= 7.071A$$

For a series-connected RLC circuit with $R = 4\Omega$ and $L = 25\,mH$, calculate:

(i) the value of C that will produce a quality factor of 50.

(ii) ω_1, ω_2 and B

(iii) The average power dissipated at $\omega = \omega_0, \omega_1$ & ω_2

Use $V_m = 100V$.

Solution:

(i) Using $Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$

$\Rightarrow \omega_0 = \frac{QR}{L} = \frac{50 \times 4}{25 \times 10^{-3}} = 8\,Krad/s$

~~~~ $Q = \frac{1}{\omega_0 RC}$

$\Rightarrow C = \frac{1}{Q\omega_0 R} = \frac{1}{50 \times 8 \times 10^3 \times 4} = \frac{0.000625}{10^3}$

$= 0.625 \times 10^{-6}\,f = 0.625\,\mu f$

(ii) ω_1, ω_2 & B:
~~~~

(i) $\omega_1 \Rightarrow \omega_0 - \frac{B}{2}$

where $B = \frac{\omega_0}{Q} = \frac{8000}{50} = 160$ rad/s

hence $\omega_1 = \omega_0 - \frac{B}{2}$

$= 8000 - \frac{160}{2} = 8000 - 80$

$\Rightarrow \omega_1 = 7,920$ rad/s

& $\omega_2 = 8000 + \frac{160}{2} = 8000 + 80$

$\omega_2 = 8,080$ rad/s

(ii) $P(\omega_0) = \frac{1}{2} \frac{V_m^2}{R}$; $V_m = 100$ V

$= \frac{1}{2} \cdot \frac{100 \times 100}{4} = \frac{10,000}{8} = 1250$

$P(\omega_0) = 1,250$ W $= 1.25$ KW

$P(\omega_1) = P(\omega_2) = \frac{V_m^2}{4R} = \frac{100 \times 100}{4 \times 4} = \frac{10,000}{16}$

$= 6250$ W $= 0.625$ KW

PARALLEL RESONANCE

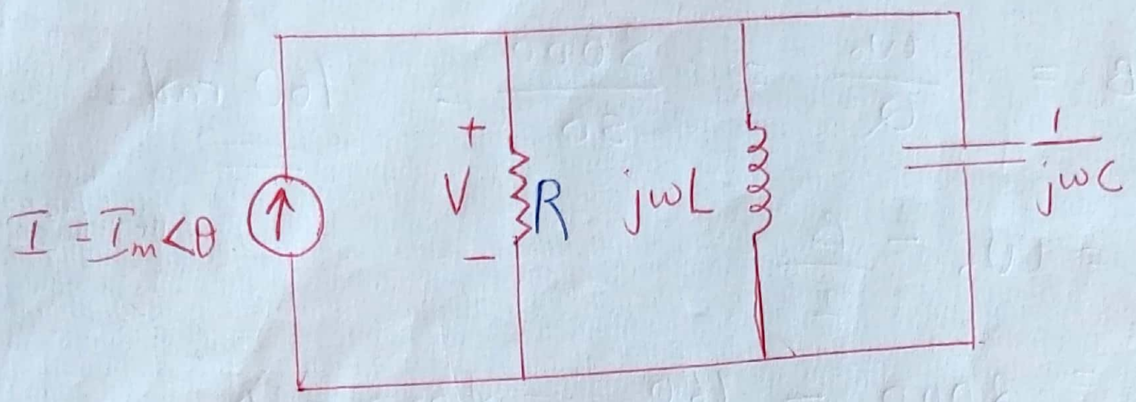


Fig: Parallel Resonant Circuit

For the circuit, the admittance, Y is:

$$Y = H(\omega) = \frac{I}{V} = \frac{1}{Z} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \quad \text{—— (i)}$$

Again, resonance occurs when the imaginary part of Y = 0 i.e

$$\omega C - \frac{1}{\omega L} = 0 \quad \text{—— (ii)}$$

$$\omega_0^2 CL = 1$$

$$\Rightarrow \omega_0 = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}} \text{ rad/s} \quad \text{—— (iii)}$$

which is the same for the ~~counterpart~~ series counterpart.

At resonance, the parallel LC combination acts like an open circuit, hence the entire current flows through R.

ω_1 and ω_2 are obtained by replacing R, L & C for the series circuit with $1/R$, C and L. And we obtain:

$$\left.\begin{aligned} \omega_1 &= -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \\ \omega_2 &= \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \end{aligned}\right\} \quad \text{— (iv)}$$

and, $B = \omega_2 - \omega_1 = \dfrac{1}{RC}$ — (v)

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L} \quad \text{— (vi)}$$

$\omega_0 C = 1/\omega_0 L$

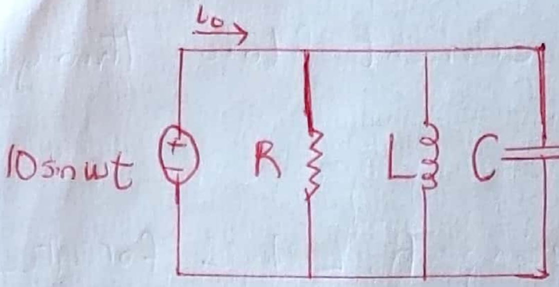
Eqs. (iv), (v) and (vi) apply to parallel circuits only.

Again, for high-Q circuits, i.e. $Q \geq 10$:

$$\omega_1 \simeq \omega_0 - \frac{B}{2} \; ; \quad \omega_2 \simeq \omega_0 + \frac{B}{2} \quad \text{— (vii)}$$

Example:

In the parallel RLC circuit here,



(i) Calculate ω_0, B and Q

(ii) find ω_1 and ω_2

(iii) Determine the power dissipated at ω_0, ω_1 & ω_2.

Take $R = 8\,K\Omega$, $L = 0.2\,mH$, $C = 8\,\mu F$

Solution

(i) $$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2\times10^{-3}\times 8\times10^{-6}}}$$

$$= \frac{1}{\sqrt{16\times10^{-1}\times10^{-9}}} = \frac{1}{\sqrt{4^2\times(10^{-5})^2}}$$

$$\Rightarrow \omega_0 = \frac{10^5}{4} = 25\,Krad/s$$

and $$Q = \frac{R}{\omega_0 L} = \frac{8\times10^3}{25\times10^3\times0.2\times10^{-3}} = \frac{8}{5\times10^{-3}}$$

or $Q = \omega_0 RC$

$$Q = \frac{8000}{5} = 1600$$

$$\Rightarrow B = \omega_0/Q = \frac{25\times1000}{1600} = 15.625$$

$$B = 15.625\,rad/s$$

Since $Q >>> 10$, the circuit is a high-Q circuit.

Hence;

$$\omega_1 \simeq \omega_0 - \frac{B}{2} = 25000 - 7.812$$

$$= 24,992 \text{ rad/s}$$

$$\omega_2 \simeq \omega_0 + \frac{B}{2} = 25,000 + 7.812$$

$$= 25,008 \text{ rad/s}$$

(iii) At $\omega = \omega_0$, $Y = 1/R$ and $Z = R = 8k\Omega$,

Then: $I_0 = V.Y = \frac{V_{max.}}{Z=R} = \frac{10\angle -90}{8,000} = 1.25\angle -90 \text{ mA}$

At resonance, the entire current flows through R, and the power dissipated at $\omega = \omega_0$ is:

$$P = \frac{1}{2}|I_0|^2 R = \frac{1}{2}(1.25\times10^{-3})^2(8\times10^3)$$

$$= 6.25 \text{ mW}$$

or $P = \frac{V_m^2}{2R} = \frac{100}{2\times8\times10^3} = 6.25 \text{ mW}$

At $\omega = \omega_1, \omega_2$;

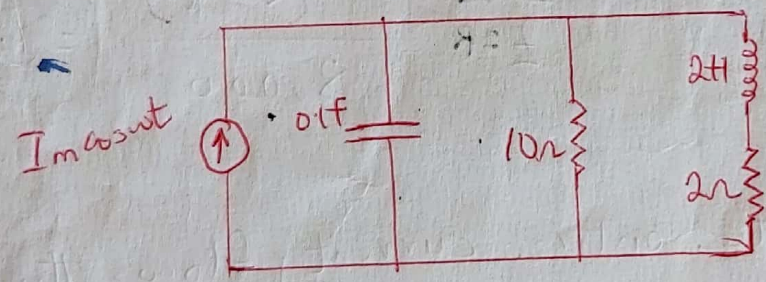
$$P = \frac{V_m^2}{4R} = \frac{10 \times 10}{4 \times 8 \times 10^3} = \underline{\underline{3.125\,mW}}$$

Exercise

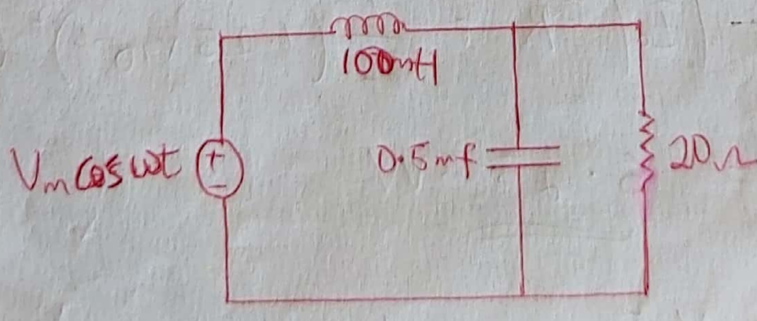
a) A parallel resonant circuit has $R = 100\,k\Omega$, $L = 20\,mH$, and $C = 5\,nF$. Calculate ω_0, ω_1, ω_2, Q and B.

Answer: 100 Krad/s, 99 Krad/s, 50, 2 Krad/s.

(ii) Determine the resonant frequency of the circuit below:

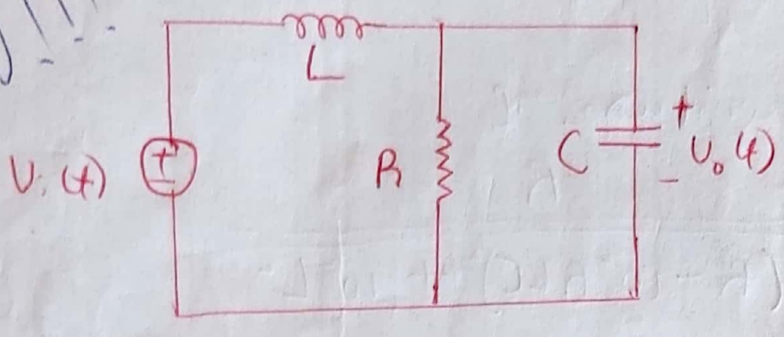


(iii) Calculate the resonant frequency of the circuit below:



Examples on Filter Design

(a) Determine the type of filter shown below. Calculate the cut-off frequency, given $R = 2\,k\Omega$, $L = 2\,H$, $C = 2\,\mu F$.



Solution

first, the transfer function is obtained as:

$$H(s) = \frac{V_o}{V_i} = \frac{R \parallel \frac{1}{sC}}{sL + R \parallel \frac{1}{sC}}$$

$$= \frac{\dfrac{R/sC}{R + \frac{1}{sC}}}{sL + \dfrac{R/sC}{R + \frac{1}{sC}}} \qquad \left[\frac{R/sC}{R + \frac{1}{sC}} = \frac{R}{(1 + sRC)} \right]$$

$$\Rightarrow H(s) = \frac{\dfrac{R}{(1+sRC)}}{sL + \dfrac{R}{(1+sRC)}} = \frac{R}{(1+sRC)sL + R}$$

$$= \frac{R}{s^2 RLC + R + sL}$$

$$H(\omega) = \frac{R}{-\omega^2 RLC + j\omega L + R} \qquad (s = j\omega)$$

By inspection, $H(0) = 1$ and $H(\infty) = 0$. Hence, it is a (second-order) lowpass filter.

H has a magnitude of :

$$|H| = \frac{R}{\sqrt{(R - \omega^2 RLC)^2 + \omega^2 L^2}}$$

At the cut-off frequency, $H = 1/\sqrt{2}$

$$\Rightarrow H = \frac{1}{\sqrt{2}} = \frac{R}{\sqrt{(R - \omega_c^2 RLC)^2 + \omega_c^2 L^2}}$$

and $$H^2 = \frac{1}{2} = \frac{R^2}{(R - \omega_c^2 RLC)^2 + \omega_c^2 L^2}$$

$$\Rightarrow 2R^2 = (R - \omega_c^2 RLC)^2 + \omega_c^2 L^2$$

$$2R^2 = R^2(1 - \omega_c^2 LC)^2 + \omega_c^2 L^2 \quad ???$$

$$2 = (1 - \omega_c^2 LC)^2 + \left(\frac{\omega_c L}{R}\right)^2$$

Given $R = 2K\Omega$, $C = 2\mu f$, and $L = 2H$

$$\Rightarrow 2 = (1 - \omega_c^2 \times 2 \times 2 \times 10^{-6})^2 + (\omega_c \times 10^{-3})^2$$

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$$\Rightarrow \quad 2 = \left[(1-4\omega_c^2)^2 + \omega_c^2\right] \times 10^{-6}$$

$$\Rightarrow \quad 16\omega_c^4 - 7\omega_c^2 - 1 = 0$$

Using Almighty formula:

$$\omega_c^2 = 0.5509 \text{ and } -0.1134$$

$$\Rightarrow \quad \omega_c = \sqrt{0.5509} = 0.7422 \times \sqrt{10^{-6}} \text{ rad/s}$$

$$= 0.7422 \times 10^{-3} \text{ rad/s}$$

~~742×10^{-6} rad/s~~

$$16(\omega_c^2)^2 - 7\omega_c^2 - 1 = 0$$

$$16x^2 - 7x - 1 = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-(-7) \pm \sqrt{49 - 4\cdot(-1)\cdot(16)}}{32}$$

$$\frac{7+\sqrt{49+64}}{32} ; \frac{7-\sqrt{\;}}{32}$$

$$\frac{7+10.63}{32} ; \frac{7-\sqrt{49+64}}{32}$$

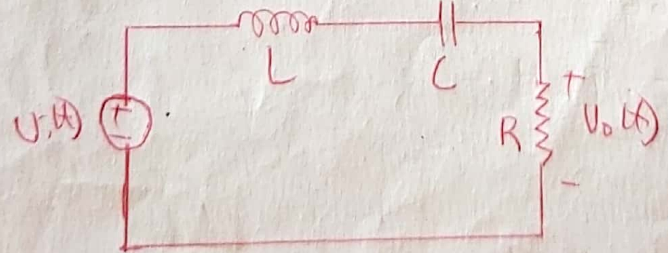
$$= \frac{17.63}{32} ; \frac{7-10.63}{32}$$

$$\omega_c^2 = 0.5509 ; \frac{-3.63}{32}$$

$$x^2 = -0.1134$$

Class Example

For the circuit below, design a bandpass filter with a lower cut-off frequency of 20.1 KHz and an upper cut-off frequency of 20.3 KHz. Take R = 20KΩ; calculate L, C and Q.



Solution

Given: $f_1 = 20.1\text{KHz}$, $f_2 = 20.3\text{KHz}$, $R = 20K\Omega$

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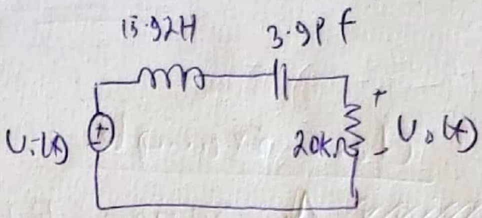
$$C = \frac{1}{(126918)^2 \times 15.92}$$

$$= \underline{3.9 pF}$$



(Test | Exam Q)

If a band stop filter rejects a 200-Hz sinusoid while passing other frequencies, calculate the values of L and C when R = 150Ω and the bandwidth is 100Hz.

Solution

Assuming its a series resonant filter circuit,

$$B = 2\pi(100) = 200\pi \text{ rad/s}$$

also, $B = \frac{R}{L}$, $\Rightarrow L = \frac{R}{B} = \frac{150}{200\pi}$

$$\Rightarrow L = \underline{0.2387H}$$

Rejecting 200-Hz implies $f_0 = 200Hz$,

thus,

$$\omega_0 = 2\pi f_0 = 2\pi \cdot 200 = 400\pi$$

and $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\Rightarrow C = \frac{1}{\omega_0^2 L} = \frac{1}{(400\pi)^2 (0.2387)}$$

$C = 2.653\mu F$

Class example

~~Design~~ A series RLC-type bandpass filter has its cut-off frequencies as 10KHz and 11KHz; Assume C = 80pF, find R, L and Q?

Solution

$f_1 = 10KHz$, $f_2 = 11KHz$, $C = 80pF$

first, convert the frequencies to ω in rad/s;

i.e. $\omega_1 = 2\pi f_1 = 2 \times \pi \times 10{,}000 = 62831.853\, rad/s$

$\omega_2 = 2\pi f_2 = 2 \times \pi \times 11{,}000 = 69115.038\, rad/s$

hence, $\omega_0 = \sqrt{\omega_2 \times \omega_1}$

$= \sqrt{62831.853 \times 69115.038} = 65898.6034$

$\omega_0 \simeq 65.90k\, rad/s$

and $B = \omega_2 - \omega_1 = 69115.038 - 62831.853$

$B = 6283.185\, rad/s$

$B \simeq 6.283k\, rad/s$

$\therefore Q = \frac{\omega_0}{B} = \frac{65.90}{6.283} = 10.488$

$Q \simeq 10.50$

Also, from $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\Rightarrow L = \frac{1}{\omega_0^2 \times C} = \frac{1}{(65898.6034)^2 \times 80\times10^{-12}}$$

$$L = \underline{\underline{2.90H}}$$

$R = ?$

From $B = R/L$, $\Rightarrow R = LB$

$$R = 2.9 \times 6283.185$$

$$= 18221.2365$$

$$R \cong \underline{\underline{18.22K\Omega}}$$

Exercise

Class Ex.

(1) Determine the range of frequencies that will be passed by a series RLC bandpass filter with $R = 10\Omega$, $L = 25mH$ and $C = 0.4\mu F$. find the quality factor, Q.

Ans: $1.54KHz$; $1.61KHz$; 25

Test / Exam Q

(2) Design a series RLC ~~reson~~ circuit with $B = 20 rad/s$, and $\omega_0 = 1,050 rad/s$. find the quality factor if $R = 10\Omega$.

Ans: $L = 0.5H$; $C = 2\mu f$ and $Q = 50$