7.7 THE EQUATION OF A CIRCLE THROUGH THREE NON-COLLINEAR POINTS

Let the equation of the circle be $x^2 + y^2 + 2gx + 2hy + c = 0$, and the three points be (x_1, y_1), (x_2, y_2), (x_3, y_3), since the circle passes through all the three points, the co-ordinates of each point must satisfy the equation of the circle. Hence

$$x_1^2 + y_1^2 + 2gx_1 + 2hy_1 + c = 0 \quad \text{.........(i)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2hy_2 + c = 0 \quad \text{.........(ii)}$$

$$x_3^2 + y_3^2 + 2gx_3 + 2hy_3 + c = 0 \quad \text{.........(iii)}$$

The three simultaneously equations can be solved for g, h and c.

Example:
Find the equation of the circle that passes through the points (6, 1), (3, 2), (2, 3)

Solution:
Let the equation of the circle be $x^2 + y^2 + 2gx + 2hy + c = 0$

Then since point (6, 1) lies on the circle

$$6^2 + 1^2 + 2g(6) + 2h(1) + c = 0$$
$$36 + 1 + 12g + 2h + c = 0$$
$$37 + 12g + 2h + c = 0 \quad \text{.........(i)}$$

Similarly (3, 2) lies on

$$3^2 + 2^2 + 2(3)g + 2(2)h + c = 0$$
$$9 + 4 + 6g + 4h + c = 0$$
$$13 + 6g + 4h + c = 0 \quad \text{.........(ii)}$$

and (2, 3) lies on

$$2^2 + 3^2 + 2(2)g + 2(3)h + c = 0$$
$$4 + 9 + 4g + 6h + c = 0$$
$$13 + 4g + 6h + c = 0 \quad \text{.........(iii)}$$

Equation (i) - 2x(ii) gives

$$37 + 12g + 2h + c = 0$$
$$\underline{26 + 12g + 8h + 2c = 0}$$

$$11 - 6h - c = 0 \quad \text{.........(iv)}$$

Again 3x(iii) - 2 x (ii) gives us

$$39 + 12g + 18h + 3c = 0$$
$$26 + 12g + 8h + 2c = 0$$

$$13 + 10h + c = 0 \quad \text{.......... (v)}$$

equation (iv) + (v) gives us

$$11 - 6h - c = 0$$
$$13 + 10h + c = 0$$

$$24 + 4h = 0$$

$$4h = -24$$
$$h = -6$$

putting h = 6 in equation (v) we have

$$13 + 10(-6) + c = 0$$
$$13 + 60 + c = 0$$
$$-47 + c = 0$$
$$c = -47$$

If we substitute h = -6 and c = 47 in equation (iii) we have

$$13 + 4g + 6(-6) + 47 = 0$$
$$60 - 36 + 4g = 0$$
$$24 + 4g = 0$$
$$4g = -24$$
$$g = -6$$

$\therefore g = -6, \; h = -6, \; c = 47$

The equation is $x^2 + y^2 - 12x - 12y + 47 = 0$

7.8 THE LENGTH OF THE TANGENT FROM A POINT P(x, y) OUTSIDE THE CIRCLE

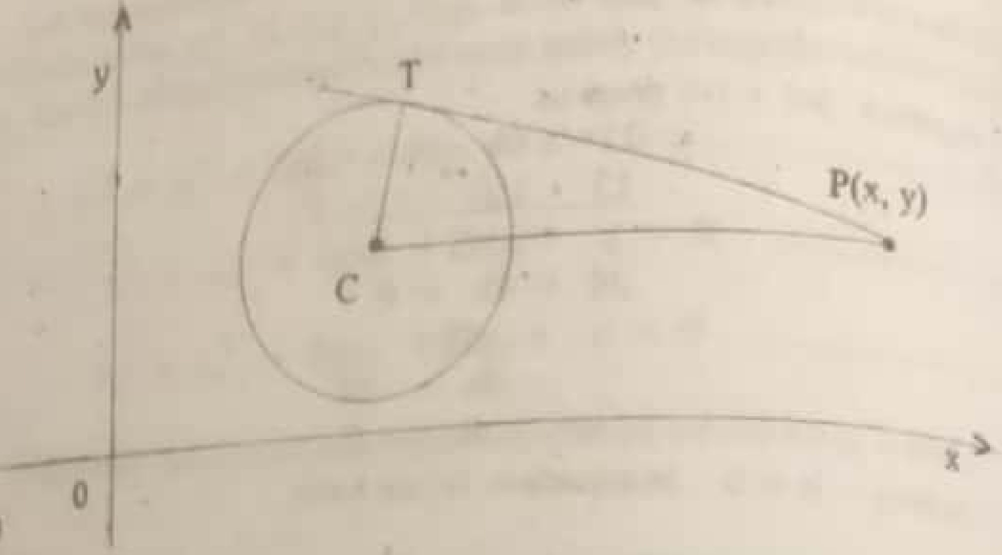

Fig 7.9

Let C be the centre of the circle and T the point of contact of the tangent PT is perpendicular to the radius TC.

Hence $PT^2 = PC^2 - CT^2$ ....(i)

Let C be the point (-g, -h) and
P be the point (x, y)

Therefore $PC^2 = (x + g)^2 + (y + h)^2$ ....(ii)

From the equation of a circle

$$X^2 + y^2 + 2gx + 2hy + c = 0$$

Applying the method of completing the squares we have

$$x^2 + 2gx + g^2 + y^2 + 2hy + h^2 + c = g^2 + h^2$$

$$(x + g)^2 + (y + h)^2 = g^2 + h^2 - c$$

Therefore the radius of a circle is given by

$$\sqrt{g^2 + h^2 - c}$$

and so $TC = \sqrt{g^2 + h^2 - c}$

$$TC^2 = g^2 + h^2 - c$$ ....(iii)

Putting equations (ii) and (iii) into equation (i) we have

$$PT^2 = (x + g)^2 + (y + h)^2 - g^2 - h^2 + c$$

$$= x^2 + y^2 + 2gx + 2hy + e$$

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Thus we obtain the square of the length of the tangent by substituting the co-ordinates of the point in the left-hand side of the equation of the circle.

Note: If PT^2 has a negative value, it indicates that P is inside the circle.

Example:
Find the length of the tangent from the point (6, 5) to the circle
$x^2 + y^2 + 2x + 4y - 21 = 0$

Solution:

$$PT^2 = 6^2 + 5^2 + 2(6) + 4(5) - 21$$
$$36 + 25 + 12 + 20 - 21 = 72$$

1.9 THE POINTS OF INTERSECTION OF THE STRAIGHT LINE $y = mx + c$ AND THE CIRCLE $x^2 + y^2 = r^2$

The co-ordinates of the points of intersection will satisfy the equation of the line and the circle simultaneously:

$$x^2 + y^2 = r^2 \quad \text{..........(i)}$$
$$y = mx + c \quad \text{..........(ii)}$$

Substituting $y = mx + c$ in equation (i) we have

$$x^2 + (mx + c)^2 = r^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = r^2 \quad \text{..........(iii)}$$
$$x^2 (1 + m^2) + 2mcx + c^2 - r^2 = 0$$

Example:
Find the point of intersection of the line $x + y - 3 = 0$ and the circle $x^2 + y^2 + x - 5y + 4 = 0$

Solution:

$$x + y - 3 = 0 \quad \text{..........(i)}$$
$$y = 3 - x \quad \text{..........(ii)}$$
$$x^2 + y^2 + x - 5y + 4 = 0$$
$$x^2 + x^2 - 6x + 9 + x - 15 + 5x + 4 = 0$$
$$2x^2 - 2 = 0$$
$$2x^2 = 2$$
$$x^2 = 2/2 = 1$$
$$x = \pm\sqrt{1} = 1 \text{ or } -1$$

Substituting $x = 1$ in equation (i) we have

$$y = x + 3$$
$$= -1 + 3$$
$$y = 2$$

Therefore one of the points of intersections is (1, 2).

Substituting $x = 1$ in equation (i) we have

$$y = -x + 3$$

$$y = -(-1) + 3$$
$$= 1 + 3$$
$$= 4$$

Therefore another point of intersection is (-1, 4).