Short Note on Rational Algebraic fractions (Before Example

We now consider the integration of rational algebraic fractions by which we mean fractions whose numerator and denominator each contain only positive integral powers of x with constant coefficients. In all cases, if the numerator is of the same/higher degree than the denominator, we first divide out. Thus, we shall have one or more terms (in x, x^2 etc or a constant) which can be immediately integrated and a fraction whose numerator is of lesser degree than the denominator.

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Integration of rational algebraic fractions (Polynomial)

* Give short note from previous note.

Examples

(1) $\displaystyle\int \frac{2x^3 - x^2 - x}{2x-3}\,dx.$

$$\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
2x-3 & 2x^3 - x^2 - x \\
 & 2x^3 - 3x^2 \\
\hline
 & 2x^2 - x \\
 & 2x^2 - 3x \\
\hline
 & 2x \\
 & 2x - 3 \\
\hline
 & 3
\end{array}$$

Which can now be written as

$$\int (x^2+x+1)\,dx + \int \frac{3}{2x-3}\,dx$$

$$\Rightarrow \frac{x^3}{3} + \frac{x^2}{2} + x + \frac{3}{2}\ln(2x-3) + c.$$

(2)

$$\int \cancel{\frac{2x^3+7x^2+2}{2x^2+x}}\,dx$$

$$\cancel{\begin{array}{r|l}
 & x + 3 \\
\hline
2x^2+x & 2x^3 + 7x^2 + 2 \\
 & 2x^3 + x^2 \\
\hline
 & 6x^2 + 2 \\
 & 6x^2 + 3x \\
\hline
 & 2 - 3x
\end{array}}$$

$$\cancel{\int (x+3)\,dx + \int \frac{2-3x}{2x^2+x}}$$

$$\int (x+3)\,dx + \int \frac{2\,dx}{2x^2+x} - \int \frac{3x}{2x^2+x}\,dx$$

Given the $\frac{d^2y}{dx^2}$ (Exple in n/G)

(2)

$$\int \frac{2x^2+11}{x^2+4}\,dx$$

first divide out to have

$$\begin{array}{r|l}
 & 2 \\
\hline
x^2+4 & 2x^2 + 11
\end{array}$$

$$\int 2\,dx + \int \frac{3}{x^2+4}\,dx$$

$$= 2x + \frac{3}{2}\arctan \frac{x}{2} + c.$$

(3)

$$\int \frac{x^2}{x-1}\,dx$$

$$\begin{array}{r|l}
 & x + 1 \\
\hline
x-1 & x^2 \\
 & x^2 - x \\
\hline
 & x \\
 & x - 1 \\
\hline
 & 1
\end{array}$$

$$\int (x+1)\,dx + \int \frac{1}{x-1}\,dx$$

$$\Rightarrow \frac{x^2}{2} + x + \ln(x-1) + c$$

(4)

$$\int \frac{2\theta - 3\theta^2}{1-\theta}\,d\theta.$$

PART 2

1) $\displaystyle\int \frac{x+3}{x^2+25}\,dx$

find the derivative of the denominator

$$\int \frac{\frac{1}{2}(2x)+3}{x^2+25}\,dx$$

$$= \int \frac{\frac{1}{2}(2x)}{x^2+25}\,dx + \int \frac{3\,dx}{x^2+25}$$

$$= \frac{1}{2}\int \frac{2x}{x^2+25}\,dx + 3\int \frac{dx}{x^2+25}$$

$$= \frac{1}{2}\log_e(x^2+25) + \frac{3}{5}\arctan(x/5) + c$$

(2)

$$\int \frac{3x+5}{x^2-6x+10}$$

$$= \int \frac{\frac{3}{2}(2x-6)+14}{x^2-6x+10}\,dx$$

$$= \int \frac{\frac{3}{2}(2x-6)\,dx}{x^2-6x+10} + \int \frac{14}{x^2-6x+10}\,dx$$

$$= \frac{3}{2}\log_e(x^2-6x+10) + 14\int \frac{dx}{x^2-6x+9+1}$$

$$14\int \frac{dx}{(x-3)^2+1^2}$$

$$\Rightarrow \frac{3}{2}\log_e(x^2-6x+10) + 14\tan^{-1}(x-3) + c$$

OTHERS

$$\int \tan^2 x\,dx$$

Recall $\sin^2$, $\tan^2 x = \sec^2 x + 1$

$$\int (\sec^2 x + 1)\,dx$$

$$= \tan x - x + C$$

Integration by part

For integrating product of simple functions such as $\int x e^x dx$, $\int t\sin t\,dt$, $\int e^{\theta}\cos\theta\,d\theta$ etc, we use the integration by part formular and which can is given as

$$\int u\,dv = uv - \int v\,du$$

<u>Guidelines for selecting u and dv</u> (There are exceptions but this is helpful)

LIATE (choose u to be the fn that comes first in this list)

L - Logarithm function e.g $\log_e x$ ($\ln x$)

I - Inverse trig functions e.g $\sin^{-1}x$, $\cos^{-1}x$, etc

A - Algebraic functions e.g x, x^2, x^3, x^4, etc

T - Trigonometric fns e.g $\sin x$, $\cos x$, etc

E - Exponential functions e.g e^{2x}, e^{3x}, etc.

<u>Example 1</u>

$$\int x\cos x\,dx$$

$u = x$, $dv = \cos x$

$\frac{du}{dx} = 1$ $\quad v = \sin x$

$$\int u\,dv = uv - \int v\,du$$
$$= x(\sin x) - \int \sin x\,dx$$
$$= x\sin x + \cos x + C$$

2

$$\int x^2\log_e x\,dx$$

$u = \log_e x$ $\quad dv = x^2$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^3}{3}$

$$\int u\,dv = uv - \int v\,du$$
$$= \log_e x\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

we have

$$\frac{x^3}{3}\log_e x - \int \frac{x^2}{3}dx$$
$$= \frac{x^3}{3}\log_e x - \frac{x^3}{9} + C.$$

3

$$\int x^3 e^x dx$$

$u = x^3$, $dv = e^x$

$du = 3x^2 dx$ $\quad v = e^x$

$$\int u\,dv = uv - \int v\,du$$
$$= x^3(e^x) - \int e^x . 3x^2 dx$$
$$= x^3e^x - \int 3x^2 e^x dx$$

$x^3e^x -$ $\begin{cases} u = 3x^2, & dv = e^x \\ du = 6x\,dx & v = e^x \end{cases}$

$\Rightarrow 3x^2(e^x) - \int e^x . 6x\,dx$

$$x^3e^x - 3x^2e^x + \int 6xe^x dx$$

$\begin{cases} u = 6x & dv = e^x \\ du = 6dx, & v = e^x \end{cases}$

$6xe^x - \int e^x . 6dx$

$$x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

(4)

Evaluate $\int e^x \sin x\,dx$

$u = \sin x, \quad dv = e^x$

$du = \cos x\,dx, \quad v = e^x$

$\int u\,dv = uv - \int v\,du$

$\sin x(e^x) - \int e^x \cos x\,dx$

$e^x \sin x - \int e^x \cos x\,dx$

$u = \cos x \qquad dv = e^x$

$du = -\sin x\,dx \qquad v = e^x$

$\cos x(e^x) - \int e^x(-\sin x)\,dx$

$e^x \cos x + \int e^x \sin x\,dx$

$e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x - \int e^x \sin x\,dx$

Let $I = \int e^x \sin x\,dx$

$I = e^x \sin x - e^x \cos x - I$

$2I = e^x \sin x - e^x \cos x$

$I = \dfrac{e^x \sin x - e^x \cos x}{2}$

Thus

$\int e^x \sin x\,dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x\right] + c$

(5)

a) $\int 2x^2 \ln x\,ds \qquad \left[\frac{2}{3}x^3\left(\ln x - \frac{1}{3}\right) + c\right]$

(b) $\int 3te^{2t}\,dt \qquad \left[\frac{3}{2}te^{2t} - \frac{3e^{2t}}{4}\right] + c$

(c) $\int x^2 \sin x\,dx$

Integration of Trig functions.

If the integrand is a product of sine or cosine of multiple angles, it may be expressed as a sum by means of the Identities

(1) $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

(2) $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

(3) $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

(4) $\sin A \sin B = \frac{-1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

Examples

(1) Evaluate $\int \sin 5x \sin x\,dx$

$A = 5x \quad B = 1x$

$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

$= \frac{-1}{2}\left[\cos 6x - \cos 4x\right]$

$\int \sin 5x \sin x\,dx = \frac{-1}{2}\int(\cos 6x - \cos 4x)$

$= \frac{-1}{2}\left[\frac{\sin 6x}{6} - \frac{\sin 4x}{4}\right]$

$= \frac{-\sin 6x}{12} + \frac{\sin 4x}{8} + c.$

(2)

$\int \sin 3x \cos x\,dx$

$= \frac{-\cos 4x}{8} - \frac{\cos 2x}{4} + c.$

(3)

a) $\int \cos 5x \cos 6x\,dx$

b) $\int \sin 7x \cos 2x\,dx$

ote

call from double-angle formulae

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad \text{——} (1)$$

And

$$\cos^2 A + \sin^2 A = 1$$

$$\cos^2 A = 1 - \sin^2 A \quad \text{——} (2)$$

$$\sin^2 A = 1 - \cos^2 A \quad \text{——} (3)$$

Put (2) in (1)

$$\cos 2A = 1 - \sin^2 A - \sin^2 A$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$2\sin^2 A = 1 - \cos 2A$$

$$\boxed{\sin^2 A = \frac{1 - \cos 2A}{2}}$$

Also, put (3) in (1)

$$\cos 2A = \cos^2 A - 1 + \cos^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\boxed{\cos^2 A = \frac{1 + \cos 2A}{2}}$$

Example 1

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right] + c$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + c.$$

Example 2

$$\int \cos^2 x \, dx = \int \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{2} \int (1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right] = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

(3)

$$\left.\begin{aligned} &\int \sin^3 x \, dx \\ &= \int \sin x \cdot \sin^2 x \, dx \\ &= \int \sin x (1 - \cos^2 x) \end{aligned}\right\} \text{Later.}$$

when one power is odd and other even

Note

The integral $\int \sin^m x \cos^n x \, dx$ can be evaluated easily, if m or n is an odd integers. If m is odd, the substitution $u = \cos x$ is used; if n is odd, the substitution $u = \sin x$ is used

Example 1

$$\int \sin^3 x \cos^2 x \, dx$$

since m is odd,

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x \Rightarrow dx = \frac{-du}{\sin x}$$

And

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$= \int \sin x \cdot \sin^2 x \cdot u^2 \cdot \frac{-du}{\sin x}$$

$$= -\int \sin^2 x \cdot u^2 du$$

$$= -\int (1 - \cos^2 x) \cdot u^2 du$$

$$= \int (u^2 - 1) u^2 du$$

$$= \int (u^4 - u^2) du$$

$$= \left[\frac{u^5}{5} - \frac{u^3}{3} \right] + c$$

$$= \frac{(\cos x)^5}{5} - \frac{(\cos x)^3}{3} + c.$$

$$\int \frac{\cos^3 x}{\sin^6 x}\,dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$$

$$\cos^2 x = 1 - \sin^2 x$$

we have

$$= \int \frac{\cos x \cdot \cos^2 x}{u^6} \cdot \frac{du}{\cos x}$$

$$= \int \frac{1-\sin^2 x}{u^6}\,du = \int \frac{1-u^2}{u^6}\,du$$

$$\int \left(\frac{1}{u^6} - \frac{u^2}{u^6}\right) du$$

$$= \int \left(u^{-6} - u^{-4}\right) du$$

$$= \frac{u^{-5}}{-5} - \frac{u^{-3}}{-3} + c$$

$$= \frac{u^{-3}}{3} - \frac{u^{-5}}{5}$$

$$= \frac{(\sin x)^{-3}}{3} - \frac{(\sin x)^{-5}}{5}$$

Example 1

$$\int \sin^2 x \cos^2 x\,dx$$

Since $\sin^2 x = 1 - \cos^2 x$

$$\int (1-\cos^2 x)\cos^2 x\,dx$$

$$\int (\cos^2 x - \cos^4 x)\,dx$$

$$\int \cos^2 x\,dx - \int \cos^4 x\,dx$$

$$\frac{x}{2} + \frac{\sin 2x}{2} - \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right] + c$$

(4)

$$\int \cos^4 x\,dx$$

Since

$$\cos^2 x = \frac{1}{2}\left[1 + \cos 2x\right]$$

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2$$

$$\frac{1}{4}\int (1+\cos 2x)^2$$

$$\frac{1}{4}\int 1 + 2\cos 2x + \cos^2 2x$$

And since

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos^2 2x = \frac{1+\cos 2(2x)}{2} = \frac{1+\cos 4x}{2}$$

we have

$$\frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2}\right) dx$$

$$\frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx$$

$$\frac{1}{4}\left[\frac{3}{2}x + \frac{2\sin 2x}{2} + \frac{\sin 4x}{8}\right] + c$$

$$= \frac{3x}{8} + \frac{\cancel{4}\sin 2x}{\cancel{8}} + \frac{\sin 4x}{32} + c$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

When both powers are even we have

Example 1

$$\Rightarrow \frac{x}{8} - \frac{\sin 4x}{32} + C$$

NB

We have the same value if we use $1 - \sin^2 x = \cos^2 x$

Example 2

$$\int \sin^2 t \cos^4 t \, dt$$

$$\int \sin^2 t \,(\cos^2 t)^2 \, dt$$

$$\int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$\int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+2\cos 2t+\cos^2 2t}{4}\right) dt$$

$$= \frac{1}{8}\int \left[1+2\cos 2t+\cos^2 2t-\cos 2t-2\cos^2 2t-\cos^3 2t\right]$$

$$= \frac{1}{8}\int \left[1+\cos 2t-\cos^2 2t-\cos^3 2t\right] dt$$

$$= \frac{1}{8}\int 1+\cos 2t-\left[\frac{1+\cos 4t}{2}\right]-\cos 2t\,(1-\sin^2 2t)\, dt$$

$$= \frac{1}{8}\int \frac{1}{2}-\frac{\cos 4t}{2}+\cos 2t \sin^2 2t \, dt$$

$$\frac{1}{8}\left[\frac{t}{2}-\frac{\sin 4t}{8}+\frac{\sin^3 2t}{6}\right]+c$$

since $\int \cos 2t \sin^2 2t \, dt$ gwas

Let $u = \sin 2t$

$$\frac{du}{dt} = 2\cos 2t$$

$$du = 2\cos 2t \, dt$$

$$\frac{du}{2} = \cos 2t \, dt$$

$$\Rightarrow \int u^2 \frac{du}{2} = \frac{1}{2}\int u^2 du$$

$$= \frac{u^3}{6}+c = \frac{\sin^3 2t}{6}+c$$

{When both powers are odd}

1

$$\int \sin x \cos^3 x \, dx$$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$du = -\sin x \, dx$$

$$-du = \sin x \, dx$$

$$\int u^3 \cdot -du = -\int u^3 du$$

$$= -\left[\frac{u^4}{4}\right]+c$$

$$\Rightarrow -\frac{\cos^4 x}{4}+c$$

2

$$\int \sin^3 x \cos^5 x \, dx$$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$du = -\sin x \, dx$$

$$dx = \frac{-du}{\sin x}$$

$$\int \sin^3 x \cdot u^5 \cdot \frac{-du}{\sin x}$$

$$-\int \sin^2 x \cdot u^5 du$$

$$-\int (1-\cos^2 x)\cdot u^5 du$$

$$-\int (1-u^2)u^5 du$$

$$\int (-u^5+u^7) du$$

$$= \frac{u^8}{8}-\frac{u^6}{6}+c$$

$$= \frac{\cos^8 x}{8}-\frac{\cos^6 x}{6}+c$$

(4)

$$\int \sin^3 x \, dx$$

Recall $\sin^2 x = 1 - \cos^2 x$

$$= \int \sin x (\sin^2 x) dx$$

$$= \int \sin x (1 - \cos^2 x) dx$$

$$\int (\sin x - \sin x \cos^2 x) dx$$

$$= \left[-\cos x + \frac{\cos^3 x}{3} \right] + C$$

(5)

$$\int \sin^5 \theta \, d\theta$$

Also, $\sin^2 \theta = 1 - \cos^2 \theta$

$$= \int \sin\theta (\sin^2\theta)^2 d\theta$$

$$= \int \sin\theta (1 - \cos^2\theta)^2 d\theta$$

$$= \int \sin\theta (1 - 2\cos^2\theta + \cos^4\theta) d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta) d\theta$$

$$= -\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + C$$

(6)

(1) $\int \sin^3 x \cos^4 x \, dx = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$

Double Integral

(1)

$$\int_0^2\left[\int_1^3 \left[xy+x^2y^3\right]dx\right]dy$$

$$\int_1^3 \left(xy+x^2y^3\right)dx$$

$$=\left[\frac{x^2}{2}y+\frac{x^3}{3}y^3\right]_1^3$$

$$=\left[\frac{9y}{2}+9y^3\right]-\left[\frac{y}{2}+\frac{y^3}{3}\right]$$

$$=4y+9y^3-\frac{y^3}{3} \Rightarrow 4y+\frac{26y^3}{3}$$

$$=\int_0^2\left(4y+\frac{26y^3}{3}\right)dy$$

$$=\left[\frac{4y^2}{2}+\frac{\overset{13}{\cancel{26}}y^4}{\underset{6}{\cancel{12}}}\right]_0^2$$

$$=\left[2y^2+\frac{13y^4}{6}\right]_0^2$$

$$=\frac{2(2)^2}{1}+\frac{13(2)^4}{6}=\frac{8}{1}+\frac{208}{6}$$

$$=\frac{256}{6}=42.67$$

(2)

(2)

$$\int_1^2\left[\int_0^3 x^2y\,dx\right]dy$$

$$\int_0^3 x^2y\,dx$$

$$\left[\frac{x^3y}{3}\right]_0^3$$

$$=\int_1^2 9y\,dy$$

$$=\left[\frac{9y^2}{2}\right]_1^2$$

$$=\frac{9(2)^2}{2}-\frac{9(1)^2}{2}$$

$$=\frac{36}{2}-\frac{9}{2}=\frac{27}{2}$$

$$=13.5$$

(3)

$$\int_1^2\left[\int_0^3 (1+8xy)dx\right]dy$$

$$\int_0^3(1+8xy)dx$$

$$\left[x+\frac{8x^2y}{2}\right]_0^3$$

$$=\int_1^2(3+36y)dy$$

$$=\left[3y+\frac{36y^2}{2}\right]_1^2$$

$$\left[3y+18y^2\right]_1^2$$

$$\left[6+\cancel{71}72\right]-\left[3+18\right]$$

$$\cancel{07}78-21=57$$

$$\int_{-2}^{3} (4-x^2)\,dx$$

$$\Rightarrow \left[4x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \left[4(3) - \frac{3^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right]$$

$$= \left[12 - 9\right] - \left[\frac{-8}{1} + \frac{8}{3}\right]$$

$$= \left[3\right] - \left[\frac{-24+8}{3}\right] = \frac{3}{1} + \frac{16}{3}$$

$$= \frac{9+16}{3} = \frac{25}{3} = 8\tfrac{1}{3}$$

(2)

END OF MAT 104 NOTE.