Short Note on Integration Of Rational Algebraic fractions (Before example

We now consider the integration of rational algebraic fractions by which we mean fractions whose numerator and denominator each contain only positive integral powers of $x$ with constant coefficients. In all cases, if the numerator is of the same/higher degree than the denominator, we first divide out. Thus, we shall have one or more terms (in $x$, $x^2$ etc or a constant) which can be immediately integrated and a fraction whose numerator is of lesser degree than the denominator.

Modes of Integration

Integration of rational algebraic fractions (polynomials)

* Start note from previous note

Examples

(1) $$\int \frac{2x^3-x^2-x}{2x-3}\,dx$$

$$\begin{array}{r|l} & x^2+x+1 \\ \hline 2x-3 & 2x^3-x^2-x \\ & 2x^3-3x^2 \\ \hline & \quad 2x^2-x \\ & \quad 2x^2-3x \\ \hline & \qquad 2x \\ & \qquad 2x-3 \\ \hline & \qquad\quad 3 \end{array}$$

Which can now be written as

$$\int (x^2+x+1)\,dx + \int \frac{3}{2x-3}\,dx$$

$$\Rightarrow \frac{x^3}{3}+\frac{x^2}{2}+x+\frac{3}{2}\ln(2x-3)+c$$

(2) $$\int \frac{2x^2+11}{x^2+4}\,dx$$

first divide out to have

$$\begin{array}{r|l} & 2 \\ \hline x^2+4 & 2x^2+11 \end{array}$$

$$\int 2\,dx + \int \frac{3}{x^2+4}\,dx$$

$$= 2x + \frac{3}{2}\arctan \frac{x}{2} + c$$

(3) $$\int \frac{x^2}{x-1}\,dx$$

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \\ & x^2-x \\ \hline & \quad x \\ & \quad x-1 \\ \hline & \qquad 1 \end{array}$$

$$\int (x+1)\,dx + \int \frac{1}{x-1}\,dx$$

$$\Rightarrow \frac{x^2}{2}+x+\ln(x-1)+c$$

(4) $$\int \frac{2\theta-3\theta^2}{1-\theta}\,d\theta$$

## PART 2

(1)

$$\int \frac{x+3}{x^2+25}\,dx$$

find the derivative of the denominator

$$\int \frac{\frac{1}{2}(2x)+3}{x^2+25}\,dx$$

$$= \int \frac{\frac{1}{2}(2x)}{x^2+25}\,dx + \int \frac{3\,dx}{x^2+25}$$

$$= \frac{1}{2}\int \frac{2x}{x^2+25}\,dx + 3\int \frac{dx}{x^2+25}$$

$$= \frac{1}{2}\log_e(x^2+25) + \frac{3}{5}\arctan(x/5) + c$$

(2)

$$\int \frac{3x+5}{x^2-6x+10}$$

$$= \int \frac{\frac{3}{2}(2x-6)+14}{x^2-6x+10}\,dx$$

$$= \int \frac{\frac{3}{2}(2x-6)\,dx}{x^2-6x+10} + \int \frac{14}{x^2-6x+10}\,dx$$

$$= \frac{3}{2}\log_e(x^2-6x+10) + 14\int \frac{dx}{x^2-6x+9+1}$$

$$14\int \frac{dx}{(x-3)^2+1^2}$$

$$\Rightarrow \frac{3}{2}\log_e(x^2-6x+10) + 14\tan^{-1}(x-3) + c$$

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# Integration by Part

For integrating product of simple functions such as $\int xe^x dx$, $\int t\sin t\,dt$, $\int e^\theta \cos\theta\, d\theta$ etc, we use the integration by part formular and which can is given as

$$\int u\,dv = uv - \int v\,du$$

Guidelines for selecting u and dv (There are exceptions but this is helpful)

LIATE (choose u to be the fn that comes first in this list)

L - Logarithm function e.g $\log_e x$ ($\ln x$)

I - Inverse trig funtions e.g $\sin^{-1}x$, $\cos^{-1}x$, etc

A - Algebraic functions e.g $x$, $x^2$, $x^3$, $x^4$, etc

T - Trigonometric fns e.g $\sin x$, $\cos x$, etc

E - Exponential funtions e.g $e^{2x}$, $e^{3x}$, etc

Example 1

$$\int x\cos x\,dx$$

$u = x$, $dv = \cos x$

$\frac{du}{dx} = 1$ $\quad v = \sin x$

$$\int u\,dv = uv - \int v\,du$$

$$= x(\sin x) - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

2

$$\int x^2 \log_e x\,dx$$

$u = \log_e x$ $\quad dv = x^2$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^3}{3}$

$$\int u\,dv = uv - \int v\,du$$

$$= \log_e x\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

we have

$$\frac{x^3}{3}\log_e x - \int \frac{x^2}{3}dx$$

$$= \frac{x^3}{3}\log_e x - \frac{x^3}{9} + C$$

3

$$\int x^3 e^x\,dx$$

$u = x^3$, $dv = e^x$

$du = 3x^2 dx$ $\quad v = e^x$

$$\int u\,dv = uv - \int v\,du$$

$$= x^3(e^x) - \int e^x \cdot 3x^2 dx$$

$$= x^3e^x - \int 3x^2 e^x dx$$

$x^3e^x - \begin{cases} u = 3x^2, & dv = e^x \\ du = 6x\,dx & v = e^x \end{cases}$

$\Rightarrow 3x^2(e^x) - \int e^x \cdot 6x\,dx$

$$x^3e^x - 3x^2e^x + \int 6xe^x dx$$

$\begin{cases} u = 6x & dv = e^x \\ du = 6dx & v = e^x \end{cases}$

$6xe^x - \int e^x \cdot 6dx$

$$x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

(4)

Evaluate $\int e^x \sin x \, dx$

$u = \sin x, \quad dv = e^x$

$du = \cos x \, dx, \quad v = e^x$

$\int u\,dv = uv - \int v\,du$

$\sin x (e^x) - \int e^x \cos x \, dx$

$e^x \sin x - \int e^x \cos x \, dx$

$u = \cos x \quad dv = e^x$

$du = -\sin x \, dx \quad v = e^x$

$\cos x (e^x) - \int e^x(-\sin x) dx$

$e^x \cos x + \int e^x \sin x \, dx$

$e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x - \int e^x \sin x \, dx$

Let $I = \int e^x \sin x \, dx$

$I = e^x \sin x - e^x \cos x - I$

$2I = e^x \sin x - e^x \cos x$

$$I = \frac{e^x \sin x - e^x \cos x}{2}$$

Thus

$$\int e^x \sin x \, dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x\right] + c$$

(5)

a) $\int 2x^2 \ln x \, dx \qquad \left[\frac{2}{3}x^3\left(\ln x - \frac{1}{3}\right)\right] + c$

b) $\int 3te^{2t} dt \qquad \left[\frac{3}{2}te^{2t} - \frac{3e^{2t}}{4}\right] + c$

c) $\int x^2 \sin x \, dx$

Integration of ...

If the integrand is a product sine or cosine of multiple angles may be expressed as a sum by means of the identities

1. $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$
2. $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$
3. $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$
4. $\sin A \sin B = \frac{-1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

Examples

1. Evaluate $\int \sin 5x \sin x \, dx$

$A = 5x \quad B = 1x$

$\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

$= \frac{-1}{2}\left[\cos 6x - \cos 4x\right]$

$\int \sin 5x \sin x \, dx = \frac{-1}{2}\int(\cos 6x - \cos 4x) dx$

$= \frac{-1}{2}\left[\frac{\sin 6x}{6} - \frac{\sin 4x}{4}\right]$

$= \frac{-\sin 6x}{12} + \frac{\sin 4x}{8} + c$

(2)

$\int \sin 3x \cos x \, dx$

$= \frac{-\cos 4x}{8} - \frac{\cos 2x}{4} + c$

(3)

a) $\int \cos 5x \cos 6x \, dx$

b) $\int \sin 7x \cos 2x \, dx$