MESH, NODAL AND LOOP ANALYSIS

In this section, we will apply our knowledge and understanding of Ohm's and Kirchoff's laws – the fundamental laws of circuit theory – to develop mesh and nodal circuit analysis. The nodal analysis is based on systematic application of Kirchoff's current law (KCL), while mesh analysis is based on systematic application of Kirchoff's voltage law (KVL). Virtually any linear circuit can be analysed using these two techniques.

NODAL ANALYSIS

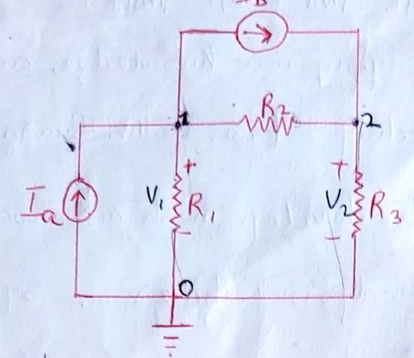
This technique involves using node voltages as the circuit variables. Steps here include:

(i) Selecting a node as the reference node (usually ground) $V=0$ and assigning $V_1, V_2, \ldots V_{n-1}$ to the remaining nodes.

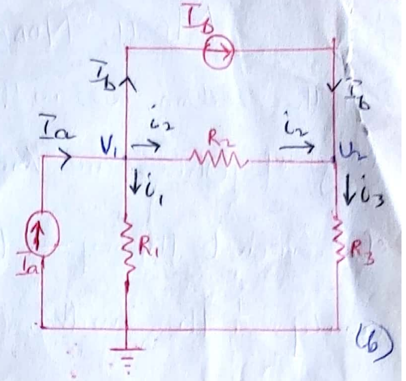
(ii) Applying KCL to each $n-1$ nonreference nodes; then using Ohm's law to express the branch currents in terms of node voltages.

(iii) Solve the equations simultaneously to obtain the unknown node voltages.

Illustration



(a)



(b)

Figs: A Typical ~~of~~ nodal circuit analysis

From (a) the ground is the reference node i.e $V = 0$.

At node 1, applying KCL:

$$I_a = I_b + i_1 + i_2$$

where $i_1 = \dfrac{V_1 - 0}{R_1}$, and $i_2 = \dfrac{V_1 - V_2}{R_2}$

$$\Rightarrow I_a = I_b + \frac{V_1}{R} + \frac{V_1 - V_2}{R_2} \quad \text{——— (i)}$$

At node 2, applying KCL:

$$i_2 + I_b = i_3$$

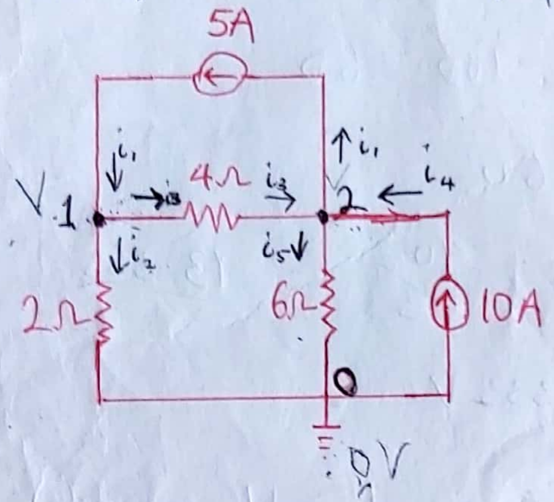
where $i_2 = \dfrac{V_1 - V_2}{R_2}$, and $i_3 = \dfrac{V_2}{R_3}$

$$\Rightarrow \frac{V_1 - V_2}{R_2} + I_b = \frac{V_2}{R_3} \quad \text{——— (ii)}$$

Equations (i) and (ii) can then be solved simultaneously or using elimination or matrix inversion or Cramer's rule.

(Test)

Examples: (i) Calculate the node voltages in the circuit shown below. Also, what current flows through the three resistors?



Solution:

Applying KCL at node 1:

$$i_1 = i_2 + i_3 \Rightarrow 5 = \frac{V_1 - V_2}{4} + \frac{V_1 - 0}{2}$$

$$\Rightarrow 20 = V_1 - V_2 + 2V_1$$

$$3V_1 - V_2 = 20 \quad \text{—— (i)}$$

Applying KCL at node 2:

$$i_3 + i_4 = i_1 + i_5$$

$$\Rightarrow \frac{V_1 - V_2}{4} + 10 = 5 + \frac{V_2 - 0}{6}$$

$$\Rightarrow 3(V_1 - V_2) + 120 = 60 + 2V_2$$

$\Rightarrow -3V_1 + 5V_2 = 60$ —— (ii)

Using elimination technique:

from (i), $V_2 = 3V_1 - 20$ —— (iii)

Substituting eq. (iii) into eq. (ii):

$\Rightarrow -3V_1 + 5(3V_1 - 20) = 60$

$-3V_1 + 15V_1 - 100 = 60$

$12V_1 = 160$

$V_1 = 160/12 = \frac{40}{3} = 13.333$

$\underline{V_1 = 13.333\ V}$

Substituting $V_1 = 13.333$ for V_1 in eq. (i):

$\Rightarrow 3(13.333) - V_2 = 20$

$40 - V_2 = 20$

$V_2 = 40 - 20 = 20$

$\underline{V_2 = 20V}$

For currents flowing through 2Ω, 4Ω, & 6Ω resistors:

$\underline{2\Omega}$: $i_2 = \frac{V_1}{2} = \frac{13.333}{2} = \underline{6.6667A}$

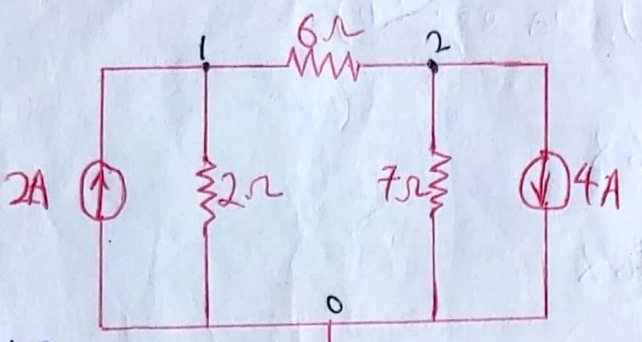
$\underline{4\Omega}$: $i_3 = \frac{V_1 - V_2}{4} = \frac{13.333 - 20}{4} = \underline{-1.6668\ A}$

$\underline{6\Omega}$: $i_5 = \frac{V_2}{6} = \frac{20}{6} = \underline{3.333A}$

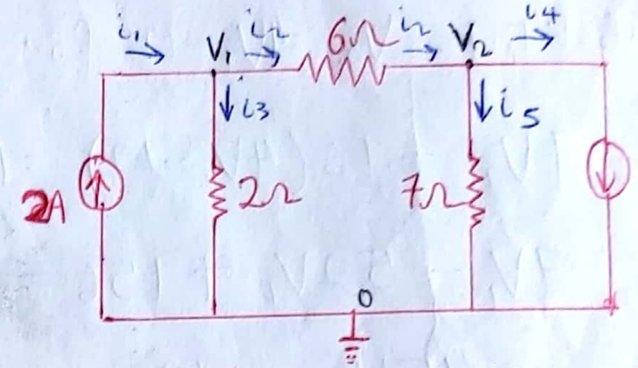
3/ The negative orientation of i_3 implies that i_3 flows in the direction opposite assumed direction.

~~class exercise~~

(ii) Obtain the node voltages and the currents flowing through the resistors in the circuit below.



(a)



(b)

Figs (a) & (b): For example above

Solution:

From (b),

Applying KCL to node 1:

$$i_1 = i_2 + i_3 \Rightarrow 2 = \frac{V_1 - V_2}{6} + \frac{V_1}{2}$$

$$\Rightarrow 12 = V_1 - V_2 + 3V_1$$

$$4V_1 - V_2 = 12 \qquad 4V_1 - V_2 = 12 \quad \text{— (i)}$$

Applying KCL to node 2:

$$i_2 = i_4 + i_5$$

$$\Rightarrow \frac{V_1 - V_2}{6} = 4 + \frac{V_2}{7}$$

$\Rightarrow 7(V_1 - V_2) = 168 + 6V_2$

$7V_1 - 13V_2 = 168$ —— (ii)

From eq(i), $V_2 = 4V_1 - 12$

substitute $V_2 = 4V_1 - 12$ for V_2 in eq. (ii)

$\Rightarrow 7V_1 - 13(4V_1 - 12) = 168$

$7V_1 - 52V_1 + 156 = 168$

$-45V_1 = 12$

$V_1 = -12/45 = -4/15\ V = -0.2667\ V$

Substituting $-4/15$ for V_1 in eq. (i)

$4(-4/15) - V_2 = 12$

$-\frac{16}{15} - V_2 = 12$

$V_2 = \frac{-16}{15} - \frac{12}{1} = \frac{-16 - 180}{15} = \frac{-196}{15}$

$V_2 = -13.0667\ V$

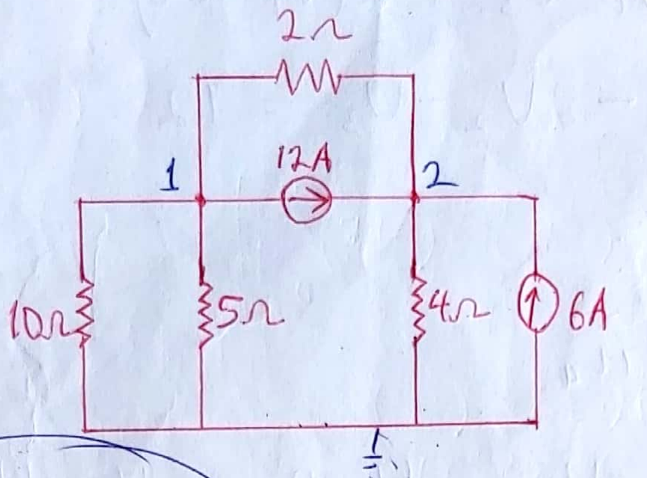
$i_3 = \frac{V_1}{2} = \frac{-0.2667}{2} = -0.13335\ A$

$i_2 = \frac{V_1 - V_2}{6} = \frac{-0.2667 + 13.0667}{6} = 2.1333\ A$

$i_5 = V_2/7 = -13.0667/7 = -1.8667\ A$

4/ Assignment ~~Exercise~~ ! ~~Class example 1~~

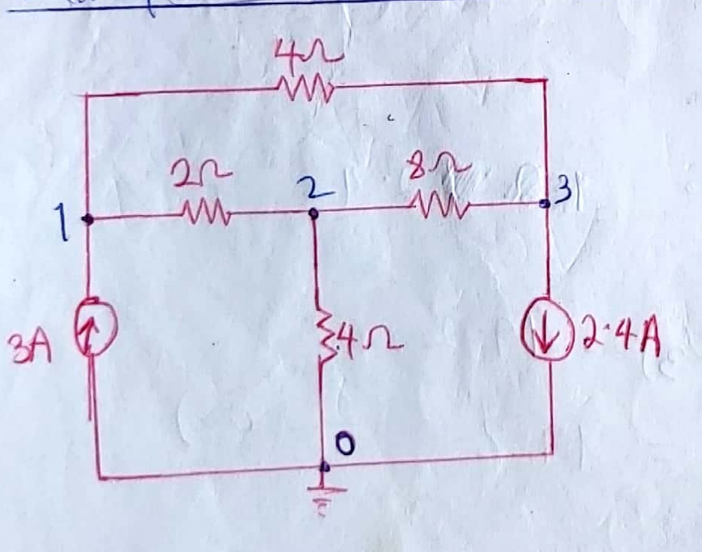
(i) Find the voltages at nodes 1 and 2 and determine the currents flowing through the four resistors in the circuit below.

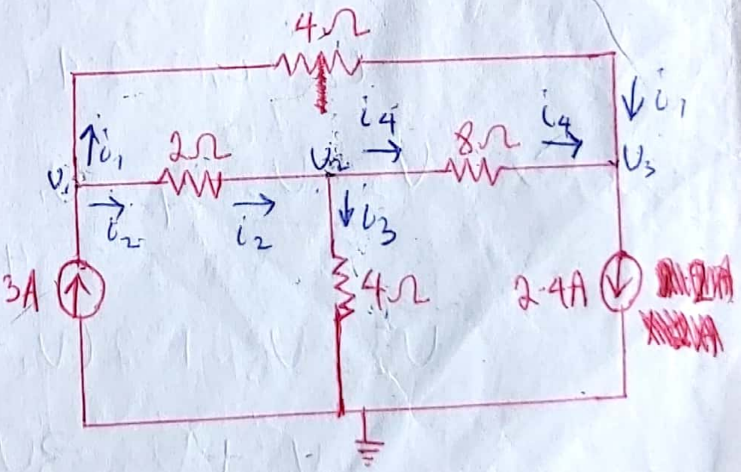


Class example 1

(ii) Obtain V_1 and V_2 and the currents through the resistors for the circuit in example (ii) if the 2A current source was replaced by a 1A current source.

Example: Circuit with 3 reference nodes (Test)





Determine the voltages at the nodes 1, 2 and 3.

At node 1,

KCL:

$$3 = i_1 + i_2 \Rightarrow 3 = \frac{V_1 - V_3}{4} + \frac{V_1 - V_2}{2}$$

$$\Rightarrow 12 = V_1 - V_3 + 2(V_1 - V_2)$$

$$3V_1 - 2V_2 - V_3 = 12 \quad \text{—— i)}$$

At node 2, KCL:

$$i_2 = i_3 + i_4$$

$$\Rightarrow \frac{V_1 - V_2}{2} = \frac{V_2 - 0}{4} + \frac{V_2 - V_3}{8}$$

$$\Rightarrow 4(V_1 - V_2) = 2V_2 + V_2 - V_3$$

$$4V_1 - 7V_2 + V_3 = 0 \quad \text{—— ii)}$$

At node 3, KCL:

$$i_4 + i_1 = 2.4$$

$$\Rightarrow \frac{V_2 - V_3}{8} + \frac{V_1 - V_3}{4} = 2.4$$

$$V_2 - V_3 + 2(V_1 - V_3) = 19.2$$

$$2V_1 + V_2 - 3V_3 = 19.2 \quad \text{—— iii)}$$

5/ Hence, we have three equations with three unknowns

Using Cramer's rule:

$$3V_1 - 2V_2 - V_3 = 12 \quad \text{--- (i)}$$

$$4V_1 - 7V_2 + V_3 = 0 \quad \text{--- (ii)}$$

$$2V_1 + V_2 - 3V_3 = 19.2 \quad \text{--- (iii)}$$

In matrix representation:

$$\begin{bmatrix} 3 & -2 & -1 \\ +4 & -7 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 19.2 \end{bmatrix}$$

$$V_1 = \frac{\Delta_1}{\Delta}; \quad V_2 = \frac{\Delta_2}{\Delta}; \quad V_3 = \frac{\Delta_3}{\Delta}$$

where $\Delta = \begin{vmatrix} 3 & -2 & -1 \\ +4 & -7 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 3\left(\overset{21-1}{+21 \cancel{+1}}\right) - (-2)\left(\overset{-14}{-12} - 2\right) + (-1)\left(+4+14\right)$

$= 3(+20) + 2(-14) - 1(18)$

$= 60 - 28 - 18 = \cancel{70}\ \underline{\underline{14}}$

$$\Delta_1 = \begin{bmatrix} 12 & -2 & -1 \\ 0 & -7 & 1 \\ 19.2 & 1 & -3 \end{bmatrix} = 12\left(\overset{20}{21-1}\right) - 0 + 19.2\left(-2-7\right)^{-9}$$

$= 240 - 172.8 = \underline{\underline{67.2}}$

$$\therefore V_1 = \Delta_1/\Delta = \frac{\cancel{336}}{\cancel{70}} = \frac{67.2}{14} = \underline{\underline{4.8\,V}}$$

~~for V_2,~~

$$\cancel{\Delta_2 = \begin{vmatrix} 3 & 12 & -1 \\ +4 & 0 & 1 \\ 2 & 19.2 & -3 \end{vmatrix} = 12(-12-2) + 0 + 19.2(3+4)}$$

$=$

for V_2:

from eq. (i), $3V_1 - 2V_2 - V_3 = 12$ —— (iv)

" " (ii), $4V_1 - 7V_2 + V_3 = 0$ —— (v)

where $V_1 = 4.8$

$\Rightarrow 3(4.8) - 2V_2 - V_3 = 12$ —— (vi)

$4(4.8) - 7V_2 + V_3 = 0$ —— (vii)

$14.4 - 2V_2 - V_3 = 12$

$V_3 = 2.4 - 2V_2$

substitute $2.4 - 2V_2$ for V_3 in eq. (vii)

$\Rightarrow 19.2 - 7V_2 + 2.4 - 2V_2 = 0$

$\equiv 21.6 - 9V_2 = 0$

$V_2 = 21.6/9 = \underline{\underline{2.4V}}$

for V_3:

from eq. (ii),

$4V_1 - 7V_2 + V_3 = 0$

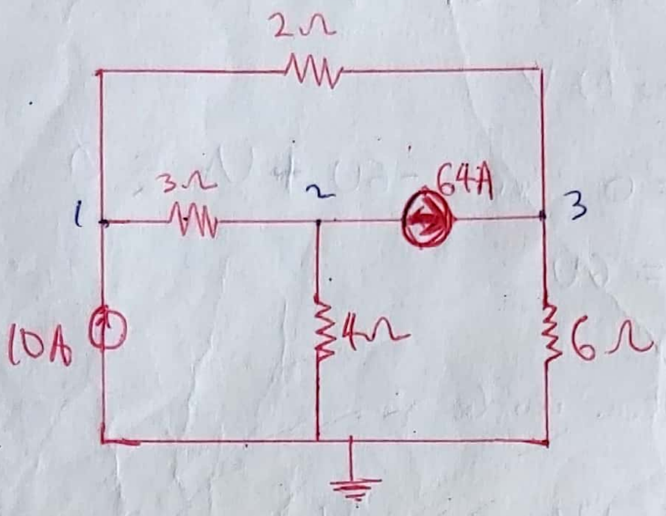
$\Rightarrow 19.2 - 16.8 = -V_3$

$V_3 = \underline{-2.4V}$

6/ Hence $U_1 = 4.8V$; $U_2 = 2.4V$, $U_3 = -2.4V$

Exercise: ~~Test~~ Classwork ✓

Find the voltages at nodes 1, 2 and 3 in the circuit below.



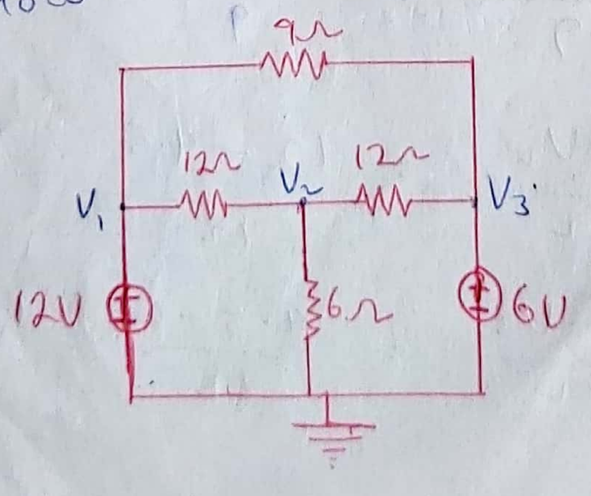
Ans. $U_1 = 80V$
$U_2 = 64V$
$U_3 = 156V$

Circuits with Independent Voltage Sources

So far, we have considered circuits with independent current sources; we now consider some examples of circuits with independent voltage sources.

Examples:

(i) Determine the node voltages and the branch currents in the circuit below.



Solution

First, the 12V and 6V independent voltage sources connected between V_1 and ground and V_3 and ground respectively simply means:

$$V_1 - 12V = 0 \quad \text{or} \quad -12V + V_1 = 0$$

$$\Rightarrow V_1 = 12V$$

and

$$V_3 - 6V = 0 \quad \text{or} \quad -6V + V_3 = 0$$

$$\Rightarrow V_3 = 6V$$

Applying KCL to the node 2:

$$\frac{V_1 - V_2}{12} = \frac{V_2 - V_3}{12} + \frac{V_2}{6}$$

$$\Rightarrow V_1 - V_2 = V_2 - V_3 + 2V_2 \equiv 12 - V_2 = V_2 - 6 + 2V_2$$

$$\Rightarrow 18 = V_2 + 2V_2 + V_2 \equiv 18 = 4V_2$$

$$V_2 = 18/4 = \underline{4.5V}$$

Currents:

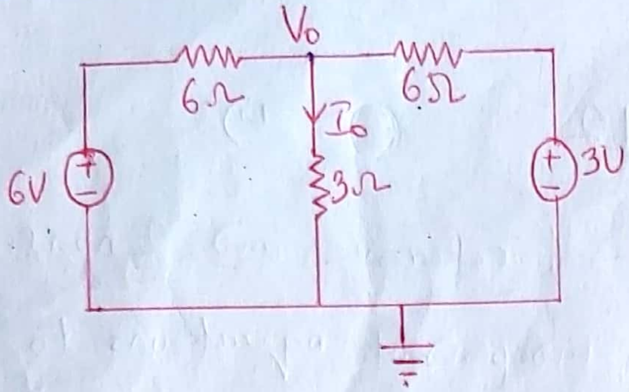
$$I_{9\Omega} = \frac{V_1 - V_3}{9} = \frac{12-6}{9} = \cancel{0.333A}\ \underline{0.667}$$

$$I_{12_1} = \frac{V_1 - V_2}{12} = \frac{12 - 4.5}{12} = \frac{7.5}{12} = \underline{0.625A}$$

$$I_{12_2} = \frac{V_2 - V_3}{12} = \frac{4.5 - 6}{12} = -1.5/12 = \underline{-0.125A}$$

7/ $I_6 = \frac{V_2}{6} = \frac{4.3}{6} = 0.73A$

Exercise



Find the current I_o in the network using nodal analysis.

Solution

At node V_o,

$$\frac{6-V_o}{6} + \frac{3-V_o}{6} = \frac{V_o}{3}$$

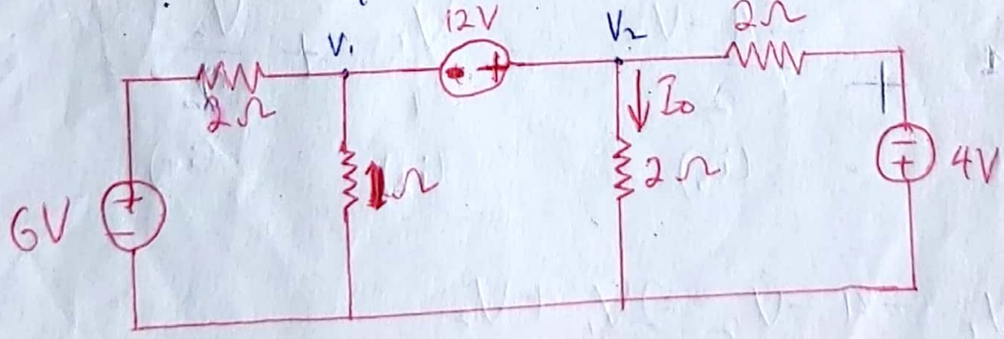
$$6-V_o+3-V_o = 2V_o$$

$$9 = 4V_o \quad V_o = 9/4\ V$$

and $I_o = \frac{V_o}{3} = \frac{9}{4} \times \frac{1}{3} = 9/12 = 3/4\ A$

A voltage source between two nodes (supernode)

Use nodal analysis to find I_o in the network below.



~~At node~~

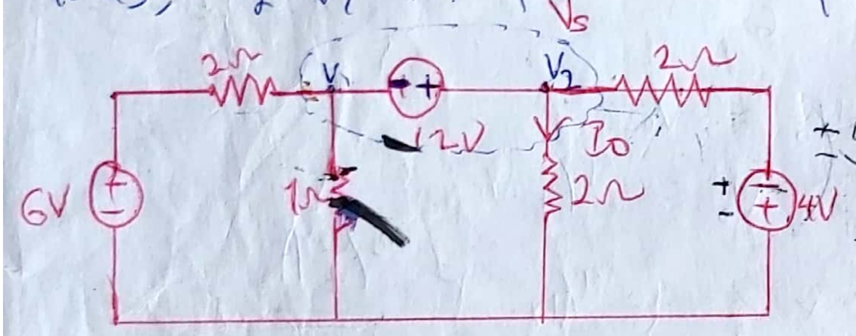
~~6~~

In this example, we have an independent voltage source (12V) connected between two nonreference nodes. The difference in potential between the two nodes is constrained by the voltage source, and hence,

$$V_2 - V_1 = 12 \longrightarrow (i)$$

Since we have 3 nodes — 1 reference and 2 nonreference, we thus require two linearly independent equations to solve the problem. The above equation is one of these.

Thus, $V_2 - V_1 = 12$ forms a supernode:



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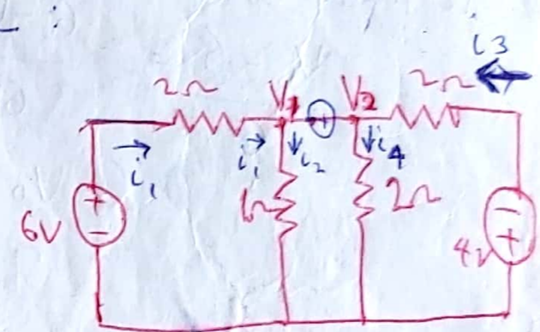


Applying KCL at the supernode:

$$i_3 + i_1 = i_2 + i_4$$

$$\Rightarrow \underset{(i_1)}{\frac{6 - V_1}{2}} + \underset{(i_3)}{\frac{-4 - V_2}{2}} = \underset{(i_2)}{\frac{V_1}{1}} + \underset{(i_4)}{\frac{V_2}{2}}$$

$$\Rightarrow 6 - V_1 - 4 - V_2 = 2V_1 + V_2$$

$$2 = 3V_1 + 2V_2 \quad \text{——} \quad (ii)$$

From (i), $V_2 = 12 + V_1$

Substituting $12+V_1$ for V_2 in eq. (ii)

$$\Rightarrow 2 = 3V_1 + 2(12+V_1)$$

$$2 = 3V_1 + 24 + 2V_1 \; ; \; 5V_1 = -22$$

$$V_1 = -22/5 \text{ V}$$

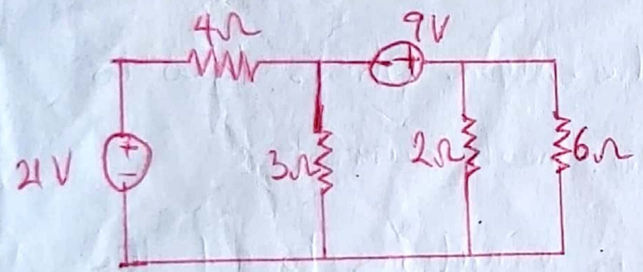
hence, $V_2 - (-22/5) = 12$

$$V_2 = \frac{12}{1} - 22/5 = \frac{60-22}{5} = 38/5 \text{ V}$$

and $I_o = V_2/2 = 38/5 \div 2 = 38/5 \times 1/2$

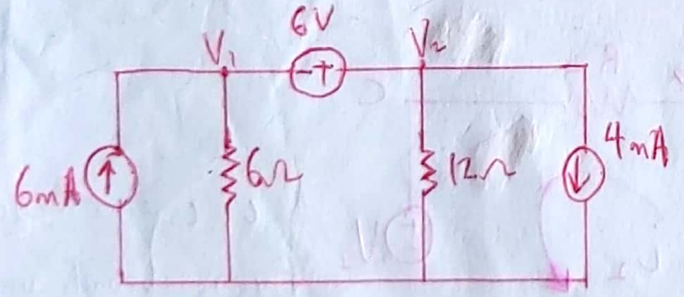
$$I_o = 38/10 = \underline{\underline{3.8 \text{ A}}}$$

Class work



Find the current through the 3Ω and 2Ω resistors.

Ans: $\underline{-0.2A}$, $\underline{4.2A}$



Find the node voltages and the currents through the 6Ω and 12Ω resistors.