DEFLECTION OF DETERMINATE STRUCTURES

1.1 GENERAL

A structure has to be stable, strong and sufficiently stiff. A stable structure will not topple, slide or roll. A strong structure will have all its parts under safe stress levels. A stiff structure will not deflect or elongate or shorten too much. Excessive deflections would make a structure functionally unsuitable.

So it is necessary to analyse structures to find their deflections as well as their stresses. Besides checking the functional acceptability, computation of deflection also serves as a first step in solving indeterminate structures.

Deflections of trusses are caused by the lengthening and shortening of its members due to tensions and compressions respectively. The displaced configurations of a truss can be determined by plotting the assemblage taking into account the deformation of its members. It can also be arrived at by considering the strain energy of the truss and the work done by the loads as they suffer displacements.

1.2 PRINCIPLE OF VIRTUAL WORK

The principle of virtual work was developed by Johann Bernoulli in 1717, and this is the most versatile of the methods available for computing deflections of structures. This method is sometimes referred to as the unit-load method. The term virtual work means the work done by a real force acting through a virtual displacement or a virtual force acting through a real displacement. The virtual work is not a real quantity but an imaginary one.

The principle of virtual work is based on the conservation of energy for a structure, which implies that work done on a structure by external loads is equal to the internal energy stored in the structure. ($U_e = U_i$).

Take a deformable structure of any shape or size and apply a series of external loads {P} to it. It will cause internal forces {F} at points throughout the structure. It is necessary that the external and internal loads be related by equations of equilibrium. As a consequence of the above loadings, external displacements Δ will occur at the P loads and internal displacements {δ} will occur at each point of internal forces {F}. These displacements do not have to be elastic, and they may not be related to the loads. However, the external and internal displacements must be related by the compatibility of the displacements. In other words, if the external displacements are known, the corresponding internal displacements are uniquely defined.

In general, the principle of virtual work and energy states that

$$\underbrace{\Sigma P \Delta}_{\text{Work of External loads}} = \underbrace{\Sigma F \delta}_{\text{Work of Internal forces}} \qquad \text{...(1.1)}$$

1.3 METHOD OF VIRTUAL FORCES

Forces and displacements are mutually interrelated. A force on a member stretching a member does work since the force moves. The member itself, because it is kept stretched by the force, stores internal energy. Obviously the internal energy stored will be equal to the work done by the force on the member.

Thus in a structure made up of several members and several external forces, there are two basic energy quantities.

(*i*) 'W_e'—*Work done by external forces as they move through some displacements (most usually in the gravity direction)*

(*ii*) 'W_i'—*Energy stored by all the strained members due to the internal forces induced in them.*

The displacement caused in simple pin jointed determinate frames can easily be determined by using the method of virtual forces, as described below in 3 steps.

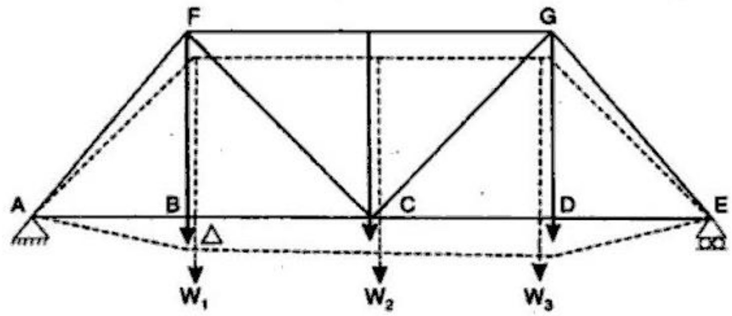


Fig. 1.1

Step 1. Suppose we want to determine the displacement at B of the truss in Fig. 1.1 due to the load system $W_1\ W_2\ W_3 = \{W\}$. We have to first solve for the internal forces $F_1\ F_2\ F_3 \ldots F_n = \{F\}$ in the members due to $\{W\}$. This is a question of statics. We know that every member will have an additional elongation (or shortening) of F_1l/AE, F_2l/AE, F_3l/AE......due to $F_1\ F_2\ F_3$........... . These member displacements will increase from 0 to $\{Fl/AE\}$ as the internal forces increase from zero to $\{F\}$.

Step 2. Now apply only a unit load (virtual force) at B in the direction of the desired deflection Δ. Find the internal forces due to the unit load ($k_1\ k_2\ k_3$ .... $= \{k\}$). We will forget about the external work done and internal energy stored due to the unit load and proceed to the next step.

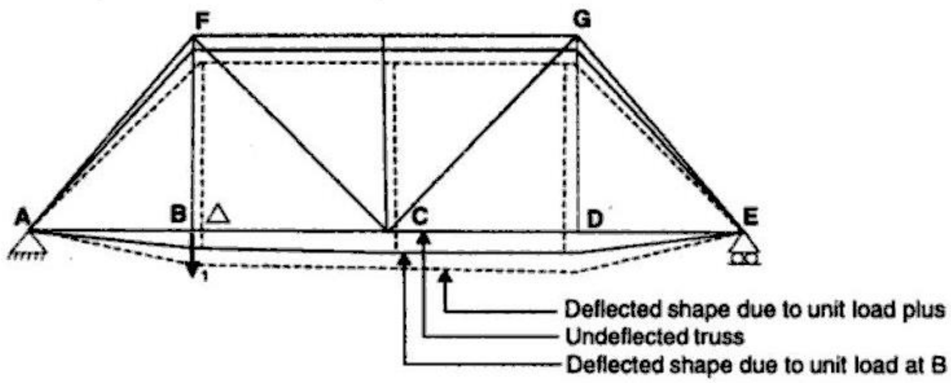


Fig. 1.2

Step 3. Now apply the {W} system in addition to the unit load, which has been applied first.

Energy Equation :

We will now have the deflected truss. The deflections are due to the unit load plus {W} system.

The external work done W_e due to the imposition of {W} is in 2 parts .

W_{e1} due to the unit load displacing through Δ at B.

W_{e2} due to the {W} system displacing progressively from zero to $\Delta_1, \Delta_2, \Delta_3$ ......

And the internal work done is also due to two causes.

W_{i1} due to the pre existing internal force system $k_1\ k_2\ k_3$ ......displacing through member displacements F_1l/AE, F_2l/AE ...... caused by {W} system

W_{i2} due to the current force system $F_1\ F_2\ F_3$ ...... displacing progressively from 0 to F_1l/AE, F_2l/AE, F_3l/AE ...... etc.

$$W_{e1} + W_{e2} = W_e = 1.\Delta + \frac{W_1\Delta_1}{2} + \frac{W_2\Delta_2}{2} \ldots\ldots$$

$$W_{i1} + W_{i2} = W_i = \Sigma \frac{kFL}{AE} + \Sigma \frac{F^2L}{2AE}$$

If {W} system were alone applied on the structure without the unit load preceeding it we would have got the relation $W_{e2} = W_{i2}$. Therefore we can infer that $1.\Delta = \Sigma \frac{kFL}{AE}$. Thus, we get the deflection Δ at B caused by the {W} system in terms of the internal forces {k} due to a unit load at B and the internal forces {F} due to the {W} system.

1.4 DEFLECTIONS OF PIN JOINTED PLANE FRAMES—PROCEDURE FOR ANALYSIS

Sign convention : Assume that tensile forces are positive and compressive forces are negative.

1. Virtual forces k. Remove all the real loads from the truss. Place a unit load on the truss at the joint and in the direction of the desired displacement. Use the method of joints or the method of sections and calculate the internal forces k in each member of the truss.

2. Real forces F. These forces are caused only by the real loads acting on the truss. Use the method of sections or the method of joints to determine the forces F in each member.

3. Virtual work equation. Apply the equation of virtual work, to determine the desired displacement.

i.e.

$$1 . \Delta = \Sigma \frac{kFL}{AE}.$$

Take proper care to retain the algebraic sign for each component k and F. If Δ turns out to be positive, then Δ is in the same direction of the unit load. If a negative value results, Δ is opposite to the direction of the unit load .

EXAMPLE 1

Determine the vertical displacement of joint C of the steel truss shown in Fig. 1.3. The cross sectional area of each member is $A = 400\ mm^2$ and $E = 2 \times 10^5\ N/mm^2$.

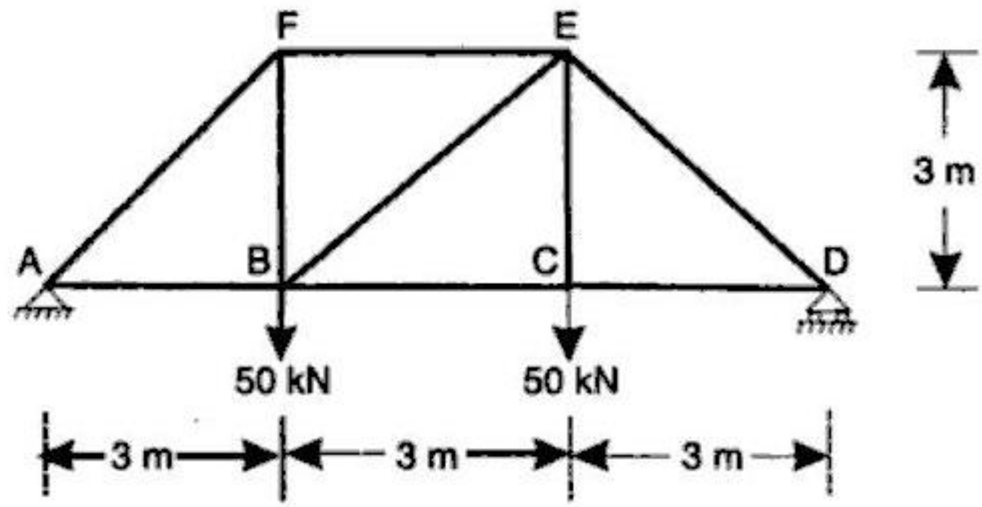


Fig. 1.3

Solution We know that $1 . \Delta = \Sigma \frac{kFL}{AE}$

Virtual forces k. Remove all the (external) loads and apply a unit vertical force at joint C of the truss. Analyse the truss using the method of joints.

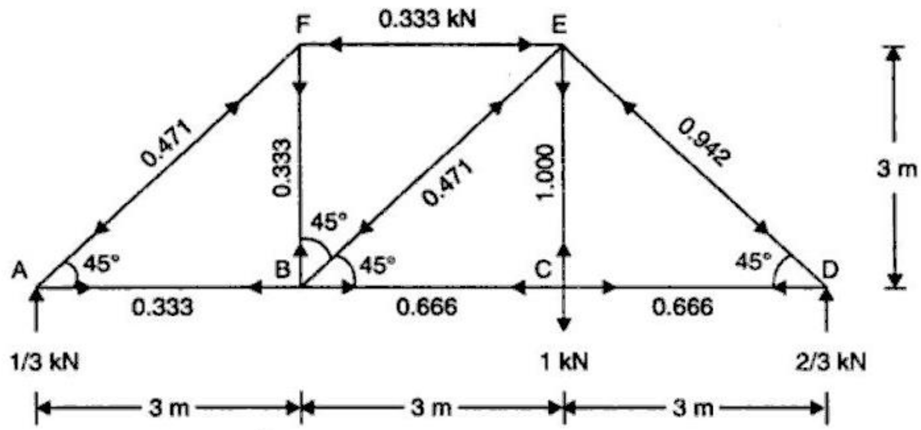


Fig. 1.4

Take moments about D, $V_A \times 9 - 1 \times 3 = 0$

$$V_A \times 9 = 1 \times 3, V_A = 1/3 \text{ kN}$$

V_D = Total load $- V_A = 1 - 1/3 = 2/3$ kN

Joint A : Initially assume all forces to be tensile.

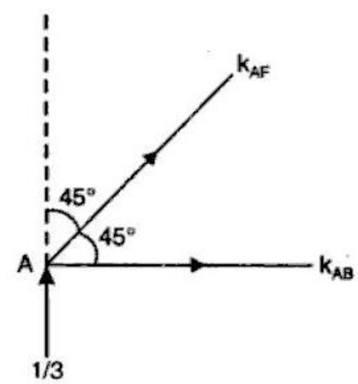
$$\Sigma V = 0 \text{ gives}$$

$$k_{AF} \cos 45° + 1/3 = 0$$
$$(\uparrow) \qquad (\uparrow)$$

$$k_{AF} \cos 45° = -1/3$$

$$k_{AF} = -\frac{1}{3 \cos 45°} = -0.471 \text{ kN}$$

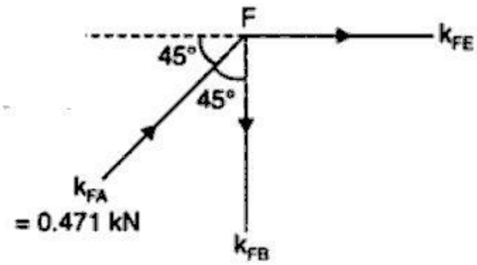
i.e. $k_{AF} = 0.471$ kN (comp.)



Joint F : $\Sigma H = 0$ gives

$$k_{FA} \cos 45° + k_{FE} = 0$$
$$(\rightarrow) \qquad (\rightarrow)$$

$$k_{FE} = -k_{FA} \cos 45° = -0.471 \times \cos 45° = -0.333 \text{ kN}$$
$$k_{FE} = 0.333 \text{ kN (comp.)}$$



$\Sigma V = 0$ gives

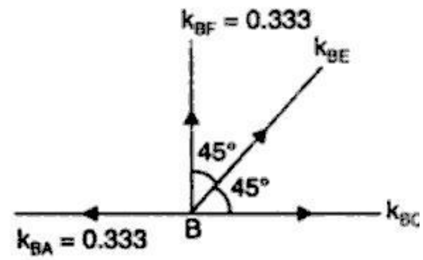
$$k_{FA} \cos 45° - k_{FB} = 0$$
$$(\uparrow) \qquad (\downarrow)$$

$$k_{FA} \cos 45° = k_{FB}$$
$$0.471 \cos 45° = k_{FB}$$

$\therefore$ $k_{FB} = 0.333$ kN (Tensile)

Joint B : $\Sigma V = 0$ gives

$$k_{BE} \cos 45° + k_{BF} = 0$$
$$(\uparrow) \qquad (\uparrow)$$



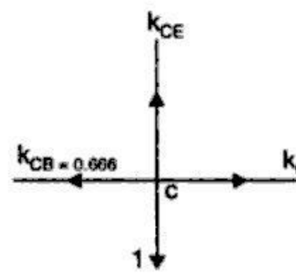
$$k_{BE} = \frac{-k_{BF}}{\cos 45°} = -\frac{0.333}{\cos 45°} = -0.471$$

$$k_{BE} = 0.471 \text{ kN (comp.)}$$

$\Sigma H = 0$ gives

$$k_{BC} - k_{BA} + k_{BE} \cos 45° = 0$$
$$(\rightarrow) \quad (\leftarrow) \qquad (\rightarrow)$$

$$k_{BC} = k_{BA} - k_{BE} \cos 45°$$
$$= 0.333 - (-0.471) \cos 45°$$
$$k_{BC} = 0.666 \text{ kN (Tensile)}$$



Joint C : $\Sigma V = 0$ gives

$$k_{CE} - 1 = 0$$
$$(\uparrow)(\downarrow)$$

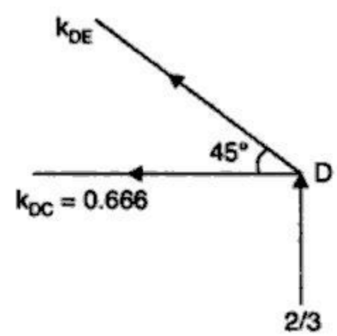
$\therefore$ $k_{CE} = 1$ kN (Tensile)

$\Sigma H = 0$ gives

$$k_{CD} - k_{CB} = 0$$
$$(\rightarrow) \quad (\leftarrow)$$

$$k_{CD} = k_{CB} = 0.666$$

$\therefore$ $k_{CD} = 0.666$ kN (Tensile)



Joint D : $\Sigma H = 0$ gives

$$k_{DE} \cos 45° + k_{DC} = 0$$
$$(\leftarrow) \qquad (\leftarrow)$$

$$\Sigma H = 0 \text{ gives}$$

$$k_{AF} \cdot \cos 45° + k_{AB} = 0$$
$$(\rightarrow) \qquad (\rightarrow)$$

$$(-0.471) \cos 45° + k_{AB} = 0$$

$$k_{AB} = 0.333 \text{ kN (Tensile)}$$

$$k_{DE} = -k_{DC}/\cos 45°$$
$$= -0.666/\cos 45°$$
$$= -0.942.$$
$$k_{DE} = 0.942 \text{ kN (comp.)}$$

Real forces, F. The real forces in the members due to the given system of external loads are calculated using the method of joints.

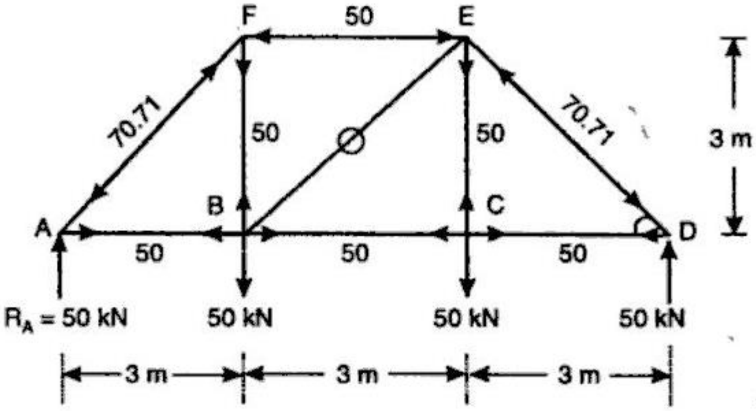


Fig. 1.5

By symmetry

$$R_A = R_D = \frac{\text{Total load}}{2} = 50 \text{ kN}$$

Joint A :

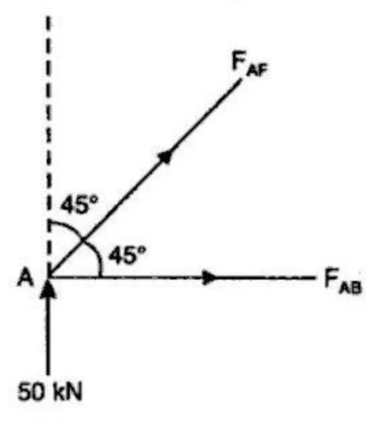
$$\Sigma V = 0 \text{ gives}$$

$$F_{AF} \cos 45° + 50 = 0$$
$$(\uparrow) \qquad (\uparrow)$$

$$F_{AF} = -50/\cos 45° = -70.71 \text{ kN}$$
$$F_{AF} = 70.71 \text{ kN (comp.)}$$
$$\Sigma H = 0 \text{ gives}$$

$$F_{AB} + F_{AF} \cos 45° = 0$$
$$(\rightarrow) \quad (\rightarrow)$$

$$F_{AB} + (-70.71) \cos 45° = 0$$
$$F_{AB} = 70.71 \cos 45°$$
$$F_{AB} = 50 \text{ kN (Tensile)}$$



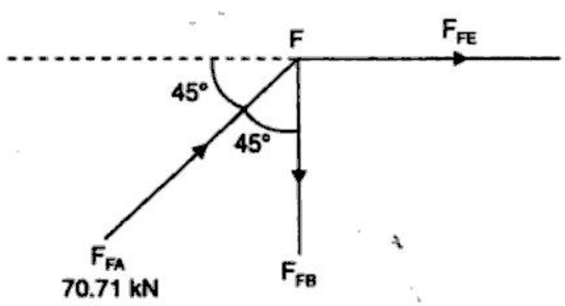
Joint F : $\Sigma V = 0$ gives

$$F_{FA} \cos 45° - F_{FB} = 0$$
$$(\uparrow) \qquad (\downarrow)$$

$$-F_{FB} = -F_{FA} \cos 45°$$
$$F_{FB} = 70.71 \cos 45°$$
$$F_{FB} = 50 \text{ kN (Tensile)}$$

$\Sigma H = 0$ gives

$$F_{FA} \cos 45° + F_{FE} = 0$$
$$(\rightarrow) \qquad (\rightarrow)$$



$$F_{FE} = -F_{FA} \cos 45°$$
$$= -70.71 \cos 45° = -50 \text{ kN}$$
$$F_{FE} = 50 \text{ kN (comp.)}$$

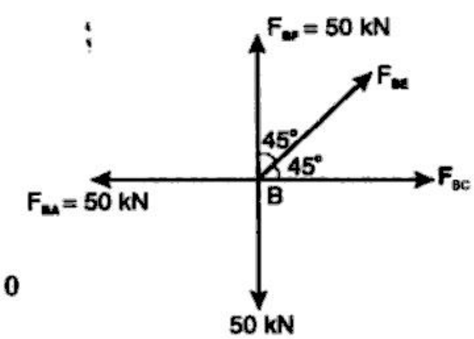
Joint B : $\Sigma V = 0$ gives

$$F_{BE} \cos 45° + F_{BF} - 50 = 0$$
$$(\uparrow) \qquad (\uparrow)(\downarrow)$$
$$F_{BE} \cos 45° = -F_{BF} + 50$$
$$= -50 + 50 = 0$$

$\therefore$ $$F_{BE} = 0$$

$\Sigma H = 0$ gives

$$F_{BC} + F_{BE} \cos 45° - F_{BA} = 0$$
$$(\rightarrow) \qquad (\rightarrow) \qquad (\leftarrow)$$
$$F_{BC} + 0 - 50 = 0 \qquad (\because \; F_{BE} = 0)$$
$$F_{BC} = 50 \text{ kN (Tensile)}$$



ASSIGNMENT

i) ***Complete the solution for joint C and D***

ii) ***Using the virtual work equation, determine the vertical deflection at joint C***