CHAPTER

3

Conservation Laws

LEARNING OUTCOMES

1. Develop equations that govern pressure variation within a static fluid
2. Determine the buoyancy forces that act on objects immersed within a fluid
3. Develop a general relationship for the time rate of change of any fluid system property
4. Apply the generalized formula for the time rate of change of a system property to the conservation of mass, conservation of momentum, and conservation of energy
5. Describe the conservation of momentum principle with acceleration
6. Derive the Navier-Stokes equations
7. Explain the Bernoulli principle and the assumptions inherent in this principle

3.1 FLUID STATICS EQUATIONS

Fluid statics problems deal with fluids that either are at rest or are only undergoing constant velocity rigid body motions. This implies that the fluid is only subjected to normal stresses because by definition a fluid will continually deform under the application of a shear stress. Shear stress would induce angular deformations within the fluid (see Figure 2.15) and therefore acceleration in particular directions. Another way to think about these types of problems is that the relative position of all fluid elements remains the same after loading. Therefore, the fluid elements would only experience pure translation or pure rotation. These types of problems fall under the class of hydrostatics and the analysis methods for these problems are typically simpler than fluid dynamics problems. Newton's second law of motion, simplified to the sum of the forces acting on the fluid is equal to zero ($\sum \vec{F} = 0$), is the primary relationship used to solve these problems.

Although fluid statics problems make the assumption that the fluid elements are not undergoing deformation, it is still possible to gain important data and insight from this type of analysis. Normal forces can be transmitted by fluids, and these forces can then be applied to devices within a biological system. For instance, by inserting a catheter into a patient, there will be some hydrostatic force that the blood transmits onto the device. While moving the catheter throughout the cardiovascular system, the hydrostatic force changes, and it may be critical to determine this force or the total force acting on the device. Imagine undergoing balloon angioplasty (in which a small balloon attached to the end of a catheter is inflated within the cardiovascular system) and not knowing the hydrostatic pressure that is being applied to the end of the catheter from the fluid. If the physician does not overcome this pressure, the balloon will not inflate and the procedure will not be completed to remedy the patient. Therefore, it is critical to understand these principles (among others) to conduct balloon angioplasty. Hydrostatic pressure is due to the weight of the fluid itself and the surrounding atmospheric pressure. Therefore, the hydrostatic pressure is different at various heights throughout the body. When a person is standing upright, the hydrostatic pressure at the top of the head is lower than that at the heart, and the hydrostatic pressure at the feet is greater than that of the heart. One way to remember this principle is when you have been standing in the same position for a long time, without moving your legs, blood pools in your lower extremities. Typically, this would eventually lead to a "cramping" feeling followed by a "pins and needles" feeling when blood is re-perfused. The reason that the blood pools in your lower limbs is that the blood in the leg cannot overcome the hydrostatic pressure to return back to the heart. Also, after sleeping, if you stand up too fast, the heart cannot overcome the new hydrostatic pressure difference and you may get light-headed. When we discuss venous return and the heart mechanics, we will show how the body can compensate for these two outcomes. Another way to recall this phenomenon is that after donating blood, the nurse will typically tell you to raise your arm. Why is this? This increases the pressure difference between your heart and your arm and will minimize the blood loss while a clot is forming at the venipuncture location. In this case, blood would have a hard time overcoming the new hydrostatic pressure gradient to enter the arm.

As stated in the previous chapter, the primary quantity of interest within fluid statics problems is the pressure field throughout the fluid. Here we will develop the equations used in fluid statics analysis. To accomplish this, Newton's second law of motion will be applied to a differential element of fluid (Figure 3.1). Recall that Newton's second law of motion is the sum of all of the forces acting on an element (body forces and surface forces) is equal to the element's mass multiplied by the element's acceleration (if density is constant). We will assume here that the only body force acting on the element is due to gravity. In most biofluid mechanics problems in this textbook, this will be the only body force that is considered. However, be cautioned that other body forces can be applied via a magnetic field (blood flow of a patient within an MRI) or by an electric field. The mass of the differential element, dm, is equal to the fluid density multiplied by the volume of the element ($dm = \rho dV = \rho dxdydz$, in Cartesian coordinates; for other coordinate systems the analysis is similar, note that $V \equiv$ volume and $v \equiv$ velocity). Therefore, the force due to gravity becomes

$$d\vec{F_b} = \vec{g}\,dm = \vec{g}\,\rho dxdydz \tag{3.1}$$

where $\vec{g}$ is the gravitational constant.

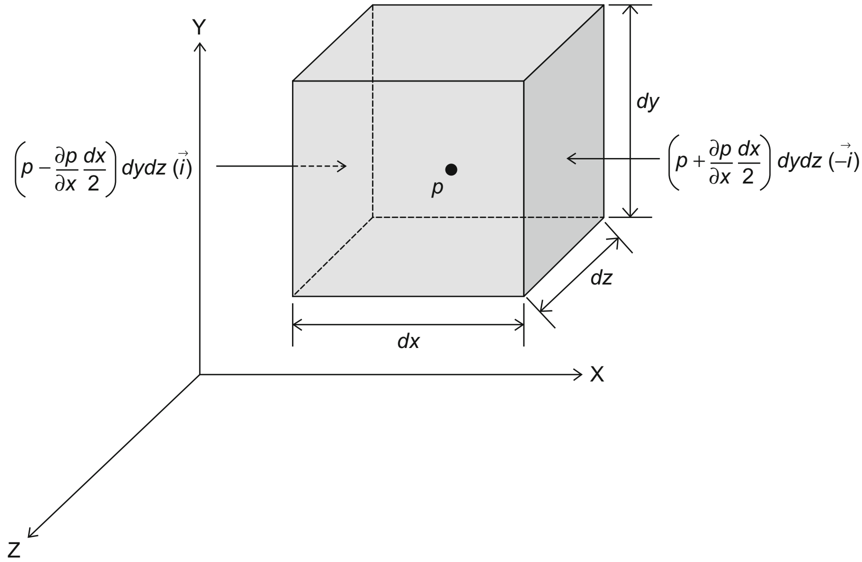

FIGURE 3.1 X-direction pressure forces that act on a differential fluid element. The same pressure forces can be derived for the other Cartesian directions.

Because the fluid is under static flow conditions, there are no shear forces applied to the fluid element. Therefore, the only surface force acting on the element is the pressure force. Pressure varies with position throughout the entire fluid. The total pressure that acts on the differential element is equal to the summation of the pressure acting on each face of the differential element. Let us define the pressure at the center of the differential element to be p (Figure 3.1). The pressure at each face would be equal to p plus or minus the particular directional pressure gradient multiplied by the distance between the center of the element and the face. For instance, in the x-direction, the pressure on the right face in the current orientation, shown in Figure 3.1, would be

$$p + \frac{\partial p}{\partial x}\frac{dx}{2}$$

whereas the pressure on the left face would be

$$p - \frac{\partial p}{\partial x}\frac{dx}{2}$$

Remember that pressure has the same unit as stress. In order to determine the force that the pressure exerts on each face, one must multiply the stress by the area over which it works (the two forces that act in the x-direction are shown in Figure 3.1). In this figure, the pressure force is also multiplied by a unit vector indicating the direction that the force acts within. Remember, for pressure there is a sign convention; a positive pressure is a compressive normal stress, and these are the forces that are shown on the differential element in Figure 3.1. To balance forces, the differential element would produce an equal and opposite force on the adjacent fluid element.

The four remaining pressure forces (that act in the y- and z-directions) can be obtained through a similar analysis method. By summing all of these six forces, the total surface force acting on the fluid element can be obtained. It would be represented as

$$\begin{aligned} d\vec{F}_s = & \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)(dydz)(\vec{i}) + \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)(dydz)(-\vec{i}) + \left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)(dxdz)(\vec{j}) \\ & + \left(p + \frac{\partial p}{\partial y}\frac{dy}{2}\right)(dxdz)(-\vec{j}) + \left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right)(dxdy)(\vec{k}) + \left(p + \frac{\partial p}{\partial z}\frac{dz}{2}\right)(dxdy)(-\vec{k}) \end{aligned} \tag{3.2}$$

Combining terms in the previous equations yields

$$\begin{aligned} d\vec{F}_s = & \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)(dydz)(\vec{i}) + \left(-p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)(dydz)(\vec{i}) + \left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)(dxdz)(\vec{j}) \\ & + \left(-p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)(dxdz)(-\vec{j}) + \left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right)(dxdy)(\vec{k}) + \left(-p - \frac{\partial p}{\partial z}\frac{dz}{2}\right)(dxdy)(\vec{k}) \\ = & -2\left(\frac{\partial p}{\partial x}\frac{dx}{2}\right)(dydz)(\vec{i}) - 2\left(\frac{\partial p}{\partial y}\frac{dy}{2}\right)(dxdz)(\vec{j}) - 2\left(\frac{\partial p}{\partial z}\frac{dz}{2}\right)(dxdy)(\vec{k}) \\ = & -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)dxdydz \end{aligned} \tag{3.3}$$

From a previous class in calculus, the final term in the parentheses (right-hand side of the equation) of Equation 3.3 is the gradient (denoted as grad or ∇) of the pressure force in Cartesian coordinates. Therefore, the surface forces acting on a differential fluid element can be simplified to

$$d\vec{F}_s = -\nabla\vec{p}\,dxdydz \tag{3.4}$$

Returning to Newton's second law, the sum of the forces acting on a differential fluid element can then be represented as

$$d\vec{F} = d\vec{F}_b + d\vec{F}_s = \vec{g}\rho dxdydz - \nabla\vec{p}\,dxdydz = (\vec{g}\rho - \nabla\vec{p})dxdydz \tag{3.5}$$

If one divides the summation of the force acting on a differential element of fluid by the unit volume (Equation 3.5), then one gets a relationship that holds for fluid particles, and is in terms of density:

$$\frac{d\vec{F}}{dxdydz} = \frac{d\vec{F}}{dV} = \vec{g}\rho - \nabla\vec{p} \tag{3.6}$$

For a static fluid flow case ($\vec{a} = 0$), Newton's second law of motion for a particle with a finite volume simplifies to

$$\frac{d\vec{F}}{dV} = \vec{g}\rho - \nabla\vec{p} = \rho\vec{a} = 0 \tag{3.7}$$

where density multiplied by the differential volume has been substituted for the differential mass. The significance of this equation is that the gravitational force must be balanced by the pressure force at each individual point within the fluid. Remember that this is only true if the fluid does not experience acceleration. In terms of the vector component equations, which must independently summate to zero, for fluid static problems, Equation 3.7 can be represented as

$$\begin{aligned} -\frac{\partial p}{\partial x} + \rho g_x = 0 \\ -\frac{\partial p}{\partial y} + \rho g_y = 0 \\ -\frac{\partial p}{\partial z} + \rho g_z = 0 \end{aligned} \tag{3.8}$$

It is conventional to choose a coordinate system in a particular way, so that the gravitational force acts in only one direction. Typically, for these types of fluids problems, the gravitational force acts in the z-direction of the Cartesian coordinate system. With this definition, Equation 3.8 simplifies to

$$\begin{aligned} \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial y} = 0 \\ \frac{\partial p}{\partial z} = -\rho g_z = \rho g \end{aligned} \tag{3.9}$$

because $g_x = g_y = 0$ and $g_z = -g$. Using these assumptions, the pressure is only a function of one coordinate variable (z) and it is independent of the other two coordinate variables (x and y). Note that pressure can act in the x/y directions; however, the pressure must be constant. The assumptions made in this analysis are that the fluid is under static flow conditions (has no acceleration term), the only body force is the gravitational force, and that gravity is only aligned with the z-axis (using the Cartesian coordinate system). Combined, this allows the use of a total derivative instead of a partial derivative in Equation 3.9. Therefore,

$$\frac{dp}{dz} = -\rho g_z \tag{3.10}$$

Equation 3.10 relates the pressure within a fluid to the vertical height of the fluid, if the assumptions made are valid or are within a reasonable estimate of the flow conditions. The previous equation can be integrated to calculate the pressure distribution throughout a static fluid, if the correct boundary conditions are applied. In general, you would need to know if the fluid's density or if gravity varies with changes in vertical distances.

Example

Calculate the static fluid pressure in the cranium at the end of systole and at the end of diastole. Assume that the cranium is 30 cm above the aortic valve and that the pressure at systole and diastole is 120 mmHg and 80 mmHg, respectively (see Figure 3.2). The density of blood is 1050 kg/m^3.

Solution

$$dp = -\rho g_z dz$$

$$\int_{p_o}^{p_1} dp = -\rho g_z \int_{z_0}^{z_1} dz$$

$$p\Big|_{p_0}^{p_1} = -\rho g_z z\Big|_{z_0}^{z_1}$$

$$p_1 - p_0 = -\rho g_z (z_1 - z_0)$$

$$p_1 = p_0 - \rho g_z (z_1 - z_0)$$

End of Systole

$$p_1 = p_0 - \rho g_z (z_1 - z_0)$$

$$p_1 = 120\ \text{mmHg} - \left(1050\,\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(30\ \text{cm} - 0\ \text{cm})\left(\frac{1\ \text{m}}{100\ \text{cm}}\right)\left(\frac{1\ \text{mmHg}}{133.32\ \text{Pa}}\right)$$

$$p_1 = 96.82\ \text{mmHg}$$

End of Diastole

$$p_1 = p_0 - \rho g_z (z_1 - z_0)$$

$$p_1 = 80\ \text{mmHg} - \left(1050\,\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(30\ \text{cm} - 0\ \text{cm})\left(\frac{1\ \text{m}}{100\ \text{cm}}\right)\left(\frac{1\ \text{mmHg}}{133.32\ \text{Pa}}\right)$$

$$p_1 = 56.82\ \text{mmHg}$$

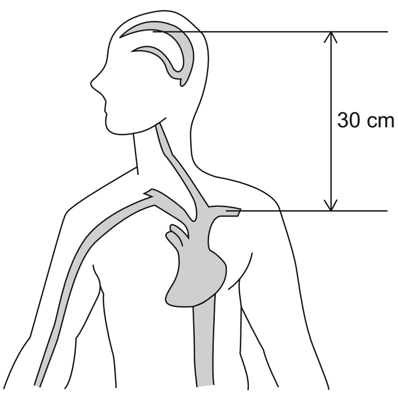

FIGURE 3.2 Difference in fluid static pressure between the aortic valve and the cranium based on height.

Notice that in the above example, the absolute change in pressure remains the same at the end of systole and diastole ($\Delta p = -23.18$ mmHg). This change in pressure is constant because we assumed that the height and the blood density did not change. Again, this is only valid if the three assumptions we made to derive this formula can be applied to the particular example.

The previous example brings us to an important distinction in fluid statics situations. All pressures must be referenced to a specific reference value. For instance, we may have chosen to call the pressure at the aortic valve 0 mmHg, and this choice would make the cranium pressure exactly −23.18 mmHg for each of the two cases in the previous example. This is true even when the aortic pressure is variable, because it is our reference and is defined as 0 mmHg at all times. There are two types of pressures that we can discuss. The first is the absolute pressure, and the second is the gauge pressure. Absolute pressure is in reference to a vacuum; this would also be the absolute (or exact) pressure of the system at your particular point of interest. Gauge pressure is the pressure of the system related to some other reference pressure, which is conventionally atmospheric pressure. Therefore, gauge pressure is actually a pressure difference and is not the actual pressure of the system. In our example above, 120 mmHg and 80 mmHg are gauge pressures. This means that the actual pressure at the aortic valve would be 120 mmHg plus 1 atm (which would be equal to an absolute pressure of 880 mmHg). In this textbook, we will refer all gauge pressures to atmospheric pressure, so that

$$p_{\text{gauge}} = p_{\text{absolute}} - p_{\text{atmospheric}}$$

Equation 3.10 describes the pressure variation in any static fluid. Changes in the pressure force are only a function of density, gravity, and the height location, assuming that the gravitational force acts only in the z-direction. In the previous example, we made the unstated assumption that changes in gravity are negligible. For most practical biofluid mechanics problems, the variation in the gravitation force with height is insignificant. Assuming that gravity is 9.81 $\mathrm{m/s^2}$ at sea level, for every kilometer above sea level gravity reduces by approximately 0.002 $\mathrm{m/s^2}$. Therefore, when the height changes that are being described are on the order of meters or centimeters (which will be typical in this textbook), the change in gravity with height can be neglected. This is a reasonable assumption. However, in some biofluid situations, it may not be a good assumption that the density is constant, so be careful applying this rule.

Remember our definition for incompressible fluids; the fluid density is constant under all conditions. In this situation, it would be appropriate to use Equation 3.10, in the form shown in the example problem. The pressure variation in a static incompressible fluid would then be

$$p = p_0 - \rho g_z (z - z_0) \tag{3.11}$$

A useful instrument to measure pressure variations solely based on height differences is a manometer. In classical fluid mechanics examples, manometers are used extensively to determine the pressure of a fluid compared to atmospheric pressure (Figure 3.3). Relating this to a biological example, manometers have been coupled to catheter systems in order to measure the intra-vascular pressure relative to atmospheric pressures. Even though the blood is flowing within the blood vessel, the blood that was diverted into the

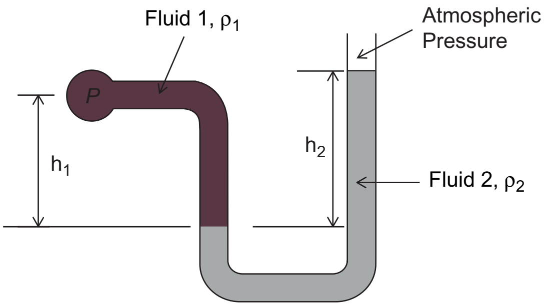

FIGURE 3.3 Schematic of a classic fluid mechanics manometer for measuring the pressure of a fluid at *P*. By measuring the differences in height, with a known open pressure, the hydrostatic pressure at *P* can be calculated.

catheter system would not be flowing, but it would maintain the same pressure of the flowing blood within the vascular system. These systems however are not very accurate when quantifying pressure because of the effects that they induce on the patient. Most likely, blood flow within the vessel would be shunted or the vessel would be ligated to insert the catheter. Therefore, the pressure that is being measured by the manometer system is not necessarily the exact physiological pressure, under normal conditions.

Example

Blood is flowing through point *P* (Figure 3.4), which is connected to a catheter tip manometer system. Blood enters the manometer and equilibrates the pressure of the various fluids within the system. Calculate the pressure within the blood vessel.

Solution

$$p_1 = p_{atm} - \rho_2 g(z_1 - z_0)$$

$$p_1 = 760\ \text{mmHg} - \left(1200\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(10\ \text{cm})\left(\frac{1\ \text{m}}{100\ \text{cm}}\right)\left(\frac{1\ \text{mmHg}}{133.32\ \text{Pa}}\right) = 751.17\ \text{mmHg}$$

$$p_2 = p_1 - \rho_1 g(z_2 - z_1)$$

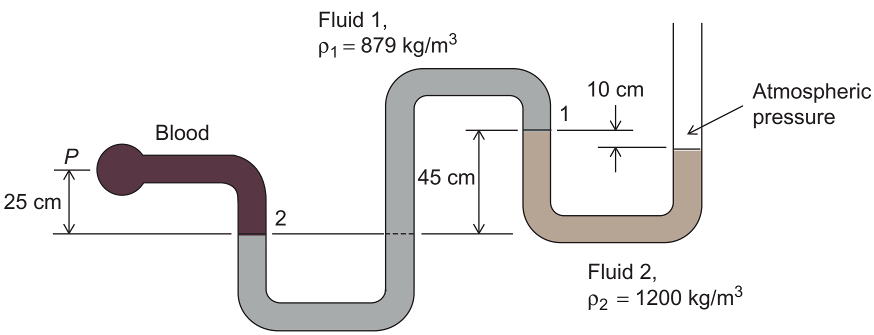

FIGURE 3.4 Schematic of a catheter tip manometer to measure intra-vascular blood pressure.

$$p_2 = 751.17\ \text{mmHg} + \left(879\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(45\ \text{cm})\left(\frac{1\ \ \text{m}}{100\ \text{cm}}\right)\left(\frac{1\ \text{mmHg}}{133.32\ \ \text{Pa}}\right) = 780.28\ \text{mmHg}$$

$$p_{blood} = p_2 - \rho_{blood}g(z_p - z_2)$$

$$p_{blood} = 780.28\ \text{mmHg} - \left(1050\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(25\ \text{cm})\left(\frac{1\ \ \text{m}}{100\ \text{cm}}\right)\left(\frac{1\ \text{mmHg}}{133.32\ \ \text{Pa}}\right) = 760.96\ \text{mmHg}$$

The following example illustrates a few principles that should be remembered when working on fluid statics problems. The first is that the pressure at an interface between two different fluids is always the same. This is how we can equate the pressure at location 1 and 2, 2 and *P*. If the fluid is continuous (same density), any location at the same height has the same pressure. Therefore, you can move around the bends without calculating each pressure change around those bends. Also, the dashed line through fluid one has the same pressure as location 2. Finally, pressure should increase as the elevation decreases and pressure should decrease as the elevation increases.

There are many biofluid problems in which density will vary. These types of fluids are compressible fluids and the density function would need to be stated within the problem. The density function would need to be given as a function of pressure and/or height. Once this function is known, then Equation 3.10 can be used to solve for the pressure distribution throughout the fluid. As an example, the density of most gases depends on the pressure and the temperature of the system. The ideal gas law represents this relationship and should be familiar to most students. The ideal gas law states that

$$p = \rho RT \tag{3.12}$$

where *R* is the universal gas constant (8.314 J/(g mol K)) and *T* is the absolute temperature (in Kelvin). The problem with using this relationship is that it introduces a new variable, *T*, into the equation, which may vary with height as well. We will typically make the assumption in this textbook that temperature fluctuations within the body can be neglected. This means that for humans, the temperature will be assumed to be 310.15 K (37°C), unless stated otherwise. Using the ideal gas law, the pressure variation in a compressible fluid, with a constant temperature is

$$\frac{dp}{dz} = -\rho g_z = \frac{p}{RT}g_z$$

$$\frac{dp}{p} = -\frac{g_z}{RT}dz$$

$$ln(p)\Big|_{p_0}^{p_1} = -\frac{g_z}{RT}z\Big|_{z_0}^{z_1}$$

$$ln(p_1) - ln(p_0) = ln\left(\frac{p_1}{p_0}\right) = -\frac{g_z}{RT}(z_1 - z_0)$$

$$\frac{p_1}{p_0} = e^{-\frac{gz}{RT}(z_1 - z_0)}$$

$$p_1 = p_0 e^{-\frac{gz}{RT}(z_1 - z_0)} \tag{3.13}$$

Depending on the particular application of the problem, the differential equation that relates pressure variations to height changes (Equation 3.10) can be solved for any fluid that has a density or temperature variation with height.

There is one important point to remember about hydrostatic pressure. Most students should be familiar with the concept that pressure gradient acts as a driving force for fluid flow (e.g., the fluid will flow from high pressure to low pressure). Hydrostatic pressure is not this driving force; otherwise, it would be easier for blood to flow from the heart to the head then from the heart to the feet (when standing upright). Hydrostatic pressure is more similar to a friction concept; that is, to move an object, enough force must be applied to overcome the frictional forces. In fluids examples, there must be enough force applied to the fluid to overcome the hydrostatic pressure gradient in order to have the fluid accelerate in that direction. Also, in some instances, hydrostatic pressure can aid in fluid movement, whereas in other instances it can hinder movement.

3.2 BUOYANCY

Buoyancy is the net vertical force acting on an object that is either floating on a fluid's surface or immersed within the fluid. To determine the net force acting on an immersed object, the same relationship for pressure variation within a static fluid can be applied. Starting from Equation 3.10, the net pressure on a three-dimensional object would need to take into account the quantity of material that is in the z-direction (Figure 3.5). Again, by taking a differential element, the net force in the z-direction would be

$$dF_z = (p + \rho g h_1)dA - (p + \rho g h_2)dA = \rho g(h_1 - h_2)dA \tag{3.14}$$

Recall from a calculus course that

$$(h_1 - h_2)dA = dV$$

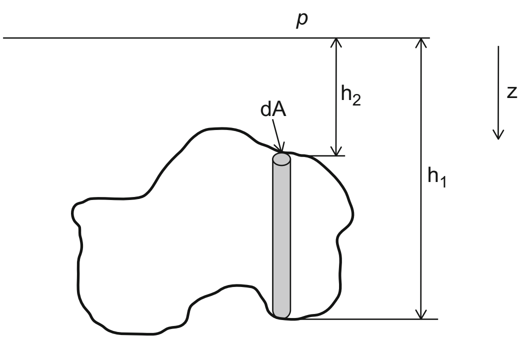

FIGURE 3.5 A body immersed in a static fluid. dA describes the cross-sectional area of the body at the location of h_1 and h_2 (which are measured in the z-axis). Various cross-sectional areas would be used to determine the buoyancy forces on an immersed object.

which is the volume of the element of interest (in Figure 3.5 this would be the shaded cylinder). Therefore, the summation of all of the forces that act on the immersed body would be

$$F_z = \int_V dF_z = \int_V \rho g dV = \rho g V$$

where V is the immersed volume of the element. This pressure force is equal to the force of gravity on the liquid displaced by the object. For biomedical applications, this is useful for designing any probe, which would be immersed within a biological fluid. Any cardiovascular implantable device would fall within this category as well.

Example

Determine the maximum buoyancy of a catheter that is inserted into the femoral artery of a patient and is passed through the cardiovascular system to the coronary artery (see Figure 3.6). The location where the catheter is inserted into the femoral artery is 50 cm below the aortic arch. The coronary artery is 5 cm below the aortic arch. Assume that the maximum buoyancy would occur at the end of systole on a normal healthy individual (120 mmHg). Also assume that the catheter is perfectly cylindrical with a diameter of 2 mm.

Solution

Pressure at incision:

$$p_1 = 120\text{ mmHg} + \left(1050\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(50\text{ cm})\left(\frac{1\text{ m}}{100\text{ cm}}\right)\left(\frac{1\text{ mmHg}}{133.32\text{ Pa}}\right) = 158.63\text{ mmHg}$$

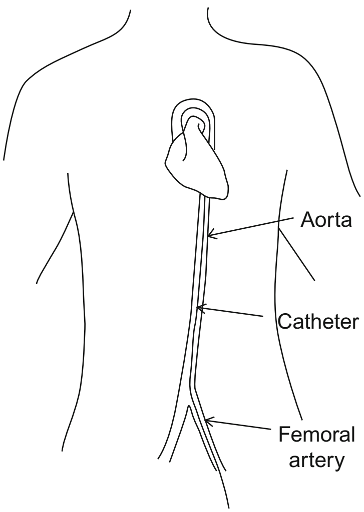

FIGURE 3.6 Catheter inserted at the femoral artery which is passed to the coronary artery. These catheters are commonly used during surgeries to remedy atherosclerotic lesions.

Pressure at coronary artery:

$$p_2 = 120 \text{ mmHg} + \left(1050\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(5 \text{ cm})\left(\frac{1 \text{ m}}{100 \text{ cm}}\right)\left(\frac{1 \text{ mmHg}}{133.32 \text{ Pa}}\right) = 123.86 \text{ mmHg}$$

Volume of catheter from femoral artery to aortic arch:

$$V = \pi\left(\frac{2 \text{ mm}}{2}\right)^2 (50 \text{ cm}) = 1.57 \text{ cm}^3$$

Volume of catheter from aortic arch to coronary artery:

$$V = \pi\left(\frac{2 \text{ mm}}{2}\right)^2 (5 \text{ cm}) = 0.157 \text{ cm}^3$$

Buoyancy force on catheter:

$$F = \left(1050\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(1.57 \text{ cm}^3) + \left(1050\frac{\text{kg}}{\text{m}^3}\right)\left(9.81\frac{\text{m}}{\text{s}^2}\right)(0.157 \text{ cm}^3) = 0.0178 \text{ N}$$

Note that the force acting on the catheter was not affected by the absolute pressure in the system because these values cancel when adding the pressure terms (see Equation 3.14).

3.3 CONSERVATION OF MASS

The previous two sections described the pressure distribution in static fluids. However, in most biofluid mechanics problems, the fluid that we are interested in will be in motion (with an acceleration component), and therefore, the previous analysis may not be applicable or may not be the most accurate. In the following four sections, we will develop relationships that govern the general fluid movement. Our analysis for each of these four sections will use a volume of interest (sometimes called a control volume) formulation, because it is normally quite difficult to identify the same mass of fluid throughout time. Remember that fluids under motion will deform, and therefore, some identifiable volume must be defined so that the laws of motion can be applied (Figure 3.7 illustrates different ways a fluid volume may be defined at different instances in time). The laws that govern a system should be familiar from earlier courses in mechanics/thermodynamics. We will extend these principles to a volume in the following formulations.

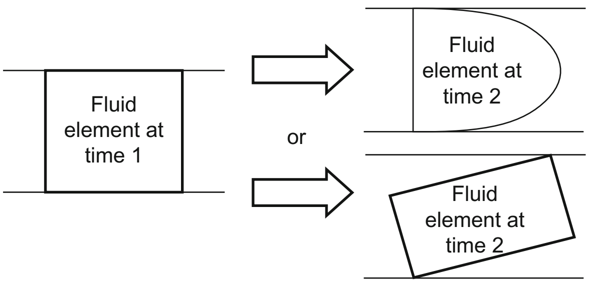

FIGURE 3.7 Two possible arrangements for a fluid element after the fluid experienced some motion. It is easier to maintain the control volume square of time 1 to analyze the fluid, instead of changing the volume of interest with time. This image shows that there are multiple possible arrangements for fluid elements after deformation, depending on the boundary conditions. It is critical during the analysis of biofluid mechanics problems to simplify these issues of deformability by choosing control volumes wisely.

First we will develop the conservation of mass principle for fluid mechanics. The basis for this principle is that mass can neither be created nor destroyed within the volume/system of interest. If the inflow mass flow rate is not balanced by the outflow mass flow rate, then there will be a change in volume or density within the volume of interest. If the inflow mass flow rate exceeds the outflow mass flow rate, mass will accumulate within the system. If the reverse scenario is true, mass will be removed from the system. Under normal conditions (e.g., not under strenuous activity), the blood volume within the heart remains constant from beat to beat. Stated in other words, the mass ejected from the aorta and the pulmonary arteries is recovered from the superior vena cava, the inferior vena cava, and the pulmonary veins. However, you can imagine a case where the residual fluid mass within the heart decreases from beat to beat. If you are experiencing severe blood loss due to a laceration, early on the heart would continue to eject the normal amount of blood, but the venous return would not be equal to this ejection volume. Therefore, the blood volume in the heart would decrease. No matter what the case is regarding mass changes within the volume of interest, mass must be conserved within the system of interest.

Before we move forward into the derivation of the conservation of mass of a system, we will derive a general relationship for the time rate of change of a system property as a function of the same property per unit mass of the volume (inherent property). This is sometimes referred to as the Reynolds Transport Theorem (RTT) formulation. For mass balance, the system property is mass and the inherent property is 1 (i.e., mass divided by mass). For balance of linear momentum, the system property is momentum ($\vec{P}$) and the inherent property is velocity ($\vec{v}$) (i.e., momentum divided by mass). For energy balance, the system property is energy, E (or entropy, S) and the inherent property is energy per unit mass, e (or entropy per unit mass, s). The system and volume of interest used in this derivation will be a cube, but this same analysis technique can be applied to any geometry (Figure 3.8). We will also assume that the shape remains the same, but this analysis holds for deformation as well. The system and volume have been chosen so that there is a region that overlaps at some later time (area 2). Mass from area 1 enters the volume of interest during Δt and mass from area 3 exits the volume of interest during Δt.

The following derivation will relate the time rate of change of any system property (W) to its inherent property (w). W and w are arbitrary properties that are only used for formula derivation (RTT). This formulation starts by using the formula of a derivative:

$$\frac{dW}{dt} = \lim_{\Delta t \to 0} \frac{W|_{t+\Delta t} - W|_t}{\Delta t}$$

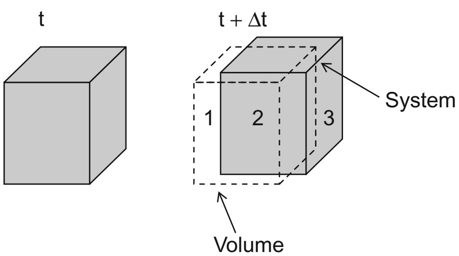

FIGURE 3.8 System and volume of interest used to derive the formula for conservation laws. The system of interest is shown by the gray shaded cube, and the volume of interest is the dashed cube. To use this formulation, one would need to know the change in time between the two states shown in this figure.

At any time t, the system is defined by the volume of interest, which keeps the same shape at all times. At time $t + \Delta t$, the system occupies area 2 and 3, instead of area 1 and 2 (for time t). Again, regardless of the area encompassed by the volume of interest, the volume (and its dimensions) remains constant at all time. Therefore, the following definitions apply for the system properties:

$$W_t = W_{volume\ of\ interest} = W_{VI}$$
$$W_{t+\Delta t} = W_2 + W_3 = W_{VI} - W_1 + W_3$$

Using these definitions in the derivative formulation

$$\frac{dW}{dt} = \lim_{\Delta t \to 0} \frac{(W_{VI} - W_1 + W_3)|_{t+\Delta t} - (W_{VI})|_t}{\Delta t} \tag{3.15}$$

which is equal to

$$\frac{dW}{dt} = \lim_{\Delta t \to 0} \frac{(W_{VI})|_{t+\Delta t} - (W_{VI})|_t}{\Delta t} + \lim_{\Delta t \to 0} \frac{(W_3)|_{t+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{(W_1)|_{t+\Delta t}}{\Delta t} \tag{3.16}$$

The first term in Equation 3.16 is equal to

$$\lim_{\Delta t \to 0} \frac{(W_{VI})|_{t+\Delta t} - (W_{VI})|_t}{\Delta t} = \frac{\partial W_{VI}}{\partial t} = \frac{\partial}{\partial t} \int_V w\rho dV \tag{3.17}$$

For the remaining two terms, a similar analysis can be conducted, to obtain

$$\begin{aligned} dW_3\big|_{t+\Delta t} &= w\rho dV\big|_{t+\Delta t} = w\rho \Delta x dA|_{t+\Delta t} \\ dW_1\big|_{t+\Delta t} &= w\rho dV\big|_{t+\Delta t} = w\rho \Delta x(-dA)|_{t+\Delta t} \end{aligned} \tag{3.18}$$

Remember that dV can be described as the change in length (i.e., from area 2 to area 3) multiplied by the differential area (in general, the cube can move in three-dimensional space). Also recall that a negative sign is included in the second term of Equation 3.18 to take care of the direction that the normal area vector is facing. The change in length can also be considered as the fluid path for any deformation that a fluid element can experience. The mass is moving to the right (Figure 3.8), but the area vector for area 1 is oriented toward the left. If we integrate the two equations in 3.18, we get

$$\begin{aligned} \int_{\text{area } 3} dW_3|_{t+\Delta t} = W_3|_{t+\Delta t} &= \int_{\text{area } 3} w\rho \Delta x dA|_{t+\Delta t} \\ \int_{\text{area } 1} dW_1|_{t+\Delta t} = W_1|_{t+\Delta t} &= \int_{\text{area } 1} w\rho \Delta x(-dA)|_{t+\Delta t} \end{aligned} \tag{3.19}$$

Substituting these values into Equation 3.16,

$$\lim_{\Delta t \to 0} \frac{(W_3)|_{t+\Delta t}}{\Delta t} = \frac{\int_{\text{area } 3} w\rho \Delta x dA}{\Delta t} = \int_{\text{area } 3} w\rho \vec{V} \Delta d\vec{A}$$

$$\lim_{\Delta t \to 0} \frac{(W_1)|_{t+\Delta t}}{\Delta t} = \frac{\int\limits_{\text{area 1}} w\rho\Delta x(-dA)}{\Delta t} = -\int\limits_{\text{area 1}} w\rho\vec{V}\Delta d\vec{A} \tag{3.20}$$

The previous equation uses the equalities for

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \vec{v}$$
$$dA = d\vec{A}$$

to change the quantities into vector form. Combining Equations 3.17 and 3.20 into Equation 3.16,

$$\frac{dW}{dt} = \frac{\partial}{\partial t}\int\limits_{V} w\rho dV + \int\limits_{\text{area 3}} w\rho\vec{v}\cdot d\vec{A} + \int\limits_{\text{area 1}} w\rho\vec{v}\cdot d\vec{A} \tag{3.21}$$

The entire system of interest consists of areas 1, 2, and 3, and we can make the assumption that there is no change in flow within region 2 during the time interval of Δt (this is why we choose to overlap the systems from the two time intervals). Therefore, $\vec{v}$ is zero for area 2, and we can combine the two area integrals in Equation 3.21 into a general form, where "area" is equal to area 1 plus area 3.

$$\frac{dW}{dt} = \frac{\partial}{\partial t}\int\limits_{V} w\rho dV + \int\limits_{\text{area}} w\rho\vec{v}\cdot d\vec{A} \tag{3.22}$$

When developing the formulation for the time rate of change of a system property, we took the limit of the system as time approached zero. This forces the relationship to be valid at the instant when the system and the control volume completely overlap. The first term of Equation 3.22 is the time rate of change of any arbitrary system property (W). The second term in Equation 3.22 is the time rate of change of the inherent property within the volume of interest (w). The third term in Equation 3.22 is the flux of the property out of the surface of interest or into the surface of interest. From this relationship, all of the conservation laws can be derived by substituting the appropriate system property and inherent property, which were described above.

In Chapter 2, we defined conservation of mass as

$$m_{\text{system}} = m_{\text{in}} - m_{\text{out}} - \frac{d(\rho V)}{dt} \tag{2.1}$$

In a more concise form, mass balance can be stated as

$$\left.\frac{dm}{dt}\right|_{\text{system}} = 0 \tag{3.23}$$

The mass of a system can be defined as

$$m_{\text{system}} = \int\limits_{m-\text{system}} dm = \int\limits_{V-\text{system}} \rho dV$$

Substituting the appropriate values for mass into Equation 3.22,

$$\left.\frac{dm}{dt}\right|_{\text{system}} = \frac{\partial}{\partial t}\int_V \rho dV + \int_{\text{area}} \rho \vec{v} \cdot d\vec{A} = 0 \tag{3.24}$$

The previous equation (Equation 3.24) describes the changes in mass within a system of interest. The first term (right-hand side of Equation 3.24), describes the time rate of change of the mass within the volume of interest. This includes any possible change in density within the volume or changes within the volume itself. The second term (right-hand side), describes the mass flux into/out of the surfaces of interest. Mass that is entering into the volume of interest would be considered a negative flux (because the velocity vector acts in an opposite direction to the area vector), whereas mass leaving the volume of interest would be a positive flux (the velocity and the area vectors are acting in the same direction). By the conservation of mass principle, the time rate of change of mass within the volume of interest has to be balanced by the flux of mass into/out of the volume of interest.

Equation 3.24 can be simplified in specific fluid cases. For an incompressible flow, there is no change in density with time/space. This simplifies Equation 3.24 to

$$\left.\frac{dm}{dt}\right|_{\text{system}} = \rho\frac{\partial V}{\partial t} + \rho\int_{\text{area}} \vec{v} \cdot d\vec{A} = 0 \tag{3.25}$$

because the volume integral of dV is simply the volume of interest. By canceling out the density terms and making a further assumption that the volume of interest does not change with time, Equation 3.25 becomes

$$\left.\frac{dm}{dt}\right|_{\text{system}} = \int_{\text{area}} \vec{v} \cdot d\vec{A} = 0 \tag{3.26}$$

A volume that does not change with time would be considered non-deformable. This is not always a good assumption in biofluids because blood vessels change shape when the heart's pressure pulse is passed through it. Also, the lungs use a shape change to drive the flow of air into or out of the system. However, in some biofluid cases, it might be acceptable to make this assumption. In this textbook, we will assume that our volume of interest is non-deformable unless stated otherwise. Equation 3.26 does not make an assumption on the flow rate (i.e., is it steady or does it change with time), so this equation is valid for any incompressible flow through a non-deformable volume. Although, remember that by definition, steady flows can have no fluid property that changes with time. Therefore, the first integral term in Equation 3.24 would be equal to zero. So for a general compressible steady flow situation, Equation 3.24 would simplify to the mass flux equation:

$$\int_{\text{area}} \rho \vec{v} \cdot d\vec{A} = 0 \tag{3.27}$$

In fluid mechanics, the integral represented in Equation 3.26 is commonly referred to as the volume (or volumetric) flow rate, Q. For an incompressible flow through a non-deformable volume, the volume flow rate into the volume must be balanced by the flow out of the volume. However, the volume flow rate can be calculated at any one location at any time within the system of interest. Its definition would be

$$Q = \int_{\text{area}} \vec{v} \cdot d\vec{A} \tag{3.28}$$

The volume flow rate divided by area is defined as the average velocity at a particular section of interest:

$$v_{\text{avg}} = \frac{Q}{A} = \frac{1}{A} \int_{\text{area}} \vec{v} \cdot d\vec{A} \tag{3.29}$$

From the special cases that we have discussed, as well as the general formula, we can now use the conservation of mass to solve various fluid mechanics problems.

Example

Determine the velocity of blood at cross-section 4 of the aortic arch schematized in Figure 3.9. Assume that the diameter of the blood vessel is 3 cm, 1.5 cm, 0.8 cm, 1.1 cm, and 2.7 cm at cross sections 1, 2, 3, 4, and 5, respectively. Branches 2, 3, and 4 make a 75°, 85°, and a 70° angle with the horizontal direction, respectively. The velocity is 120 cm/s, 85 cm/s, 65 cm/s, and 105 cm/s at 1, 2, 3, and 5, respectively. There is inflow at 1 and outflow at all of the remaining locations. Assume steady-flow at this particular instant in time and that the volume of interest is non-deformable.

Solution

Figure 3.10 highlights the given geometric constraints in this problem. The gray dashed box on this figure represents one of the possible choices for the volume of interest. We will also make the assumption that blood density does not change and has a value of 1050 kg/m^3.

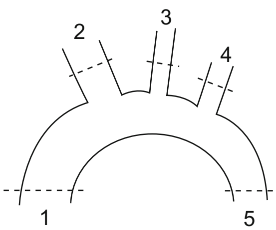

FIGURE 3.9 Schematic of the aortic arch.

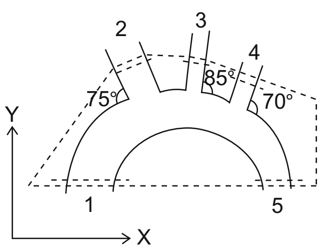

FIGURE 3.10 Figure associated with the in-text example.

We can directly apply Equation 3.26 because we have made the assumption that the density does not change with time, that the volume is not deformable, and that the flow is steady:

$$\int_{\text{area}} \vec{v} \cdot d\vec{A} = 0 \tag{3.26}$$

Equation 3.26 will further simplify to the volume flow rate at each location because of the same assumptions. Therefore,

$$-v_1 A_1 + v_2 A_2 + v_3 A_3 + v A_4 + v_5 A_5 = 0$$

The area at each location is

Location	Area
1	$\pi\left(\frac{3\ \text{cm}}{2}\right)^2 = 7.069\ \ \text{cm}^2$
2	$\pi\left(\frac{1.5\ \text{cm}}{2}\right)^2 = 1.767\ \ \text{cm}^2$
3	$\pi\left(\frac{0.8\ \text{cm}}{2}\right)^2 = 0.503\ \ \text{cm}^2$
4	$\pi\left(\frac{1.1\ \text{cm}}{2}\right)^2 = 0.950\ \ \text{cm}^2$
5	$\pi\left(\frac{2.7\ \text{cm}}{2}\right)^2 = 5.726\ \ \text{cm}^2$

Substituting the known values into the previous equation,

$$v_4 = \frac{v_1 A_1 - v_2 A_2 - v_3 A_3 - v A_5}{A_4}$$

$$v_4 = \frac{(120\,\text{cm}/s)(7.069\ \text{cm}^2) - (85\,\text{cm}/s)(1.767\ \text{cm}^2) - (65\,\text{cm}/s)(0.503\ \text{cm}^2) - (105\,\text{cm}/s)(5.726\ \text{cm}^2)}{0.950\ \text{cm}^2}$$

$$= 67.54 \frac{\text{cm}}{\text{s}}$$

This velocity would flow at a 75° angle off the positive x-axis. The inflow at location 1 was negative because the velocity vector acts in an opposite direction to the area normal vector (at all other locations they act in the same direction). To simplify this procedure, if it is known that the flow is inflow, you can assume that it has a negative sign associated with this term, whereas outflow can be assumed to have a positive sign. Density was not used in any of the formulations because it would cancel out in each term. This problem also illustrates that it does not matter which direction (x or y) the velocity is acting, because all of the mass needs to be conserved.

Example

Calculate the time rate of change in air density during expiration. Assume that the lung (see Figure 3.11) has a total volume of 6000 mL, the diameter of the trachea is 18 mm, the air flow velocity out of the trachea is 20 cm/s, and the density of air is 1.225 kg/m^3. Also assume that lung volume is decreasing at a rate of 100 mL/s.

Solution

Starting from Equation 3.24,

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_{\text{area}} \rho \vec{v} \cdot d\vec{A} = 0$$

Assume that at the instant in time that we are measuring the system, density is uniform within the volume of interest. This allows us to remove density from within the first integral:

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_{\text{area}} \rho \vec{v} \cdot d\vec{A} = 0 \tag{3.24}$$

$$\frac{\partial}{\partial t}\rho\int_V dV + \int_{\text{area}} \rho \vec{v} \cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}(\rho V) + \rho \vec{v} A = 0$$

Using the chain rule,

$$V\frac{\partial \rho}{\partial t} + \rho\frac{\partial V}{\partial t} = -\rho \vec{v} A$$

Solving this equation with the known values,

$$\frac{\partial \rho}{\partial t} = \frac{-\rho \vec{v} A - \rho \frac{\partial V}{\partial t}}{V}$$

$$= \frac{-(1.225\ \text{kg/m}^3)(20\ \text{cm/s})\left(\pi * \left(\frac{18\ \text{mm}}{2}\right)^2\right) - (1.225\ \text{kg/m}^3)(-100\ \text{mL/s})}{6000\ \text{mL}} = 0.01\frac{\text{kg}}{\text{m}^3\text{s}}$$

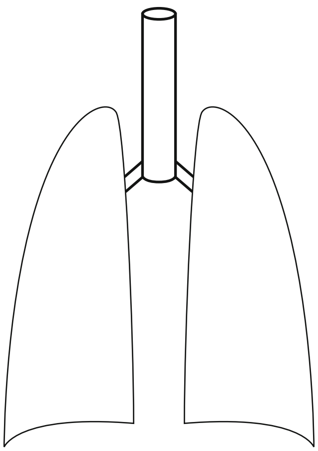

FIGURE 3.11 Schematic of the lung.

3.4 CONSERVATION OF MOMENTUM

Newton's second law of motion can be written in terms of linear momentum, which is its most general form:

$$\vec{F} = \frac{d(\vec{P})}{dt} \tag{3.30}$$

For this analysis, we want to develop a relationship for the linear momentum within a volume of interest. We will follow a similar technique as that used to develop a relationship for the conservation of mass within a volume of interest. As we alluded to, prior to developing Equation 3.22, we know the system property ($\vec{P}$) and its inherent property ($\vec{v}$), but we will take some space to define linear momentum and how it relates to fluid mechanics.

Linear momentum is defined as

$$\vec{P} = \int_{\text{system mass}} \vec{v}\,dm = \int_V \vec{v}\rho dV \tag{3.31}$$

for a volume of interest. Also, recall from an earlier discussion that the summation of the forces that act on a fluid element must include all body forces (denoted as $\vec{F}_b$) and all surface forces (denoted as $\vec{F}_s$). Physically, linear momentum is a force of motion, which is conserved unless other forces are applied to the system. By substituting the system property and the inherent property into Equation 3.22, we can get the formulation for conservation of linear momentum:

$$\frac{dP}{dt} = \frac{\partial}{\partial t}\int_V \vec{v}\rho dV + \int_{\text{area}} \vec{v}\rho\vec{v}\cdot d\vec{A} \tag{3.32}$$

Using Newton's relationship for momentum, Equation 3.32 can be represented as

$$\frac{dP}{dt} = \vec{F} = \vec{F}_b + \vec{F}_s = \frac{\partial}{\partial t}\int_V \vec{v}\rho dV + \int_{\text{area}} \vec{v}\rho\vec{v}\cdot d\vec{A} \tag{3.33}$$

Equation 3.33 states that the summation of all forces acting on a volume of interest is equal to the time rate of change of momentum within the control volume and the summation of momentum entering or leaving through the surface of interest. To solve conservation of momentum problems, the first step will be to define the volume of interest and surfaces of interest and label all of the forces that are acting on this system. This also applies when you choose to define a coordinate system that is either aligned with or not aligned with the majority of the forces; you will still need to define all forces and how they relate to the chosen coordinate axis (remember the example of a block sliding down an incline from Chapter 2). If the standard Cartesian coordinate system is chosen, then gravity aligns with one of the axes, and typically gravity will be the only body force that acts on the system. Surface forces are due to externally applied loads and are normally denoted through a pressure acting on the system. The generalized surface force will be represented as

$$\vec{F}_s = \int_{\text{area}} -p\,d\vec{A}$$

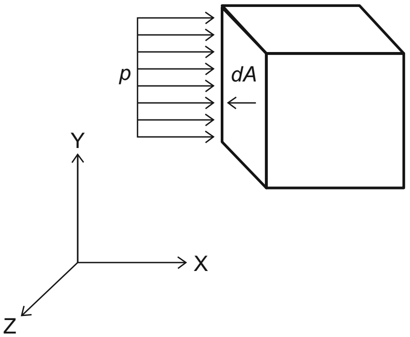

FIGURE 3.12 Pressure force acting on a surface of interest. Recall that the area vector for this surface would act in the negative x-direction, whereas the pressure forces are acting in the positive x-direction.

The negative sign in this formulation is added to maintain the sign convention for the forces acting on the system (see Figure 3.12). In Figure 3.12, the pressure is positive, but because the pressure and the area vectors act in opposite directions, their vector product would be a negative force, which does not correspond with the positive directions chosen for the coordinate system.

Unlike the conservation of mass formula, the formulas derived for the conservation for linear momentum are vector equations (compare with aortic arch example for conservation of mass). Equation 3.33 written in component form is:

$$\begin{aligned} F_x = F_{bx} + F_{sx} &= \frac{\partial}{\partial t}\int_V u\rho dV + \int_{\text{area}} u\rho\vec{v}\cdot d\vec{A} \\ F_y = F_{by} + F_{sy} &= \frac{\partial}{\partial t}\int_V v\rho dV + \int_{\text{area}} v\rho\vec{v}\cdot d\vec{A} \\ F_z = F_{bz} + F_{sz} &= \frac{\partial}{\partial t}\int_V w\rho dV + \int_{\text{area}} w\rho\vec{v}\cdot d\vec{A} \end{aligned} \tag{3.34}$$

where u, v, and w are the velocity components in the x-, y- and z-directions, respectively. As before, the product of $\rho\vec{v}\cdot d\vec{A}$ is a scalar whose sign depends on the directions of the normal area vector and the velocity vector. If these two vectors act in the same direction, the product of the vectors is positive; if they act in opposite directions, then the product is negative. Remember that the velocity vector ($\vec{v}$) in this product is not a component of velocity but is the entire velocity vector. In scalar notation, the entire form of the product would be represented as $\pm|\rho\nu^2 Acos\alpha|$, where α is defined by the coordinate system of choice and the positive or negative sign is defined through the velocity/normal area vectors relationship and the direction of the velocity component. This angle appears for the u, v, and w directional velocities. However, remember that the product of $\vec{v}\rho\vec{v}\cdot d\vec{A}$ is a vector, and the sign of this product depends on the coordinate system chosen (this defines the velocity vector sign) and the sign of the scalar. The application of these sign conventions will become apparent in some of the example problems. To determine the sign of the momentum flux through a surface, first determine the sign associated with $|\rho\nu Acos\alpha|$, and then determine the sign of each velocity component (u, v, and w). By knowing the signs of these two parts, the sign of the overall product can be determined.

Example

Determine the force required to hold the brachial artery in place during peak systole (Figure 3.13). Assume at the inlet the pressure is 100 mmHg and at the outlet the pressure is 85 mmHg (these are gauge pressures). The diameter of the brachial artery is 18 mm at the inflow and 16 mm at the outflow. The blood flow velocity at the inlet is 65 cm/s. For simplicity, neglect the weight of the blood vessel and the weight of the blood within the vessel.

Solution

Figures 3.13 and 3.14 depict what is known about the situation. The problem statement asks to solve for F_x and F_y.

To solve this problem, assume that there is steady flow at the instant in time that we are interested in, that 1 atm = 760 mmHg, that the blood vessel does not move and is not deformable, and that the flow is incompressible. First, we will need to solve for the outflow velocity using the equations for conservation of mass:

$$\left.\frac{dm}{dt}\right|_{\text{system}} = \frac{\partial}{\partial t}\int_V \rho dV + \int_{\text{area}} \rho\vec{v}\cdot d\vec{A} = 0 \tag{3.24}$$

$$\int_{\text{area}} \rho\vec{v}\cdot\left|d\vec{A}\right| = 0 \to \rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

$$\frac{(1050\ \ \text{kg/m}^3)(65\ \text{cm/s})\left(\pi\left(\frac{18\ \text{mm}}{2}\right)^2\right)}{(1050\ \ \text{kg/m}^3)\left(\pi\left(\frac{16\ \text{mm}}{2}\right)^2\right)} = v_2 = 82.27\ \text{cm/s}(-\hat{j})$$

Note that the velocity accelerates due to the step down nature of the geometry. This is not representative of what occurs in physiology, but this problem illustrates how to use conservation of mass and momentum together.

Solve for the x-component of the force needed to hold the brachial artery in place:

$$F_x = F_{bx} + F_{sx} = \frac{\partial}{\partial t}\int_V u\rho dV + \int_{\text{area}} u\rho\vec{v}\cdot d\vec{A}$$

$$F_{bx} = 0, u_2 = 0$$

$$F_{sx} = p_{\text{inflow}}A_{\text{inflow}} + p_{\text{atm}}A_1 - p_{\text{atm}}(A_{\text{inflow}} + A_1) + F_x = A_{\text{inflow}}(p_{\text{inflow}} - p_{\text{atm}}) + F_x$$

$$A_{\text{inflow}}(p_{\text{inflow}} - p_{\text{atm}}) + F_x = \int_{\text{area}} u\rho\vec{v}\cdot d\vec{A} = u_1 * (-\rho v_{\text{inflow}}A_{\text{inflow}})$$

Note that the u component of the velocity is positive, but the flux is negative because the velocity vector and the normal area vector act in opposite directions.

$$F_x = -u\rho v A_{\text{inflow}} - A_{\text{inflow}}(p_{\text{inflow}} - p_{\text{atm}}) = -u\rho v_{\text{inflow}} A_{\text{inflow}} - A_{\text{inflow}}(p_{\text{inflow}-\text{gauge}})$$

$$F_x = -(65\ \text{cm/s})(1050\ \ \text{kg/m}^3)(65\ \text{cm/s})\left(\pi\left(\frac{18\ \text{mm}}{2}\right)^2\right) - \left(\pi\left(\frac{18\ \text{mm}}{2}\right)^2\right)(100\ \text{mmHg}) = -3.5\ \ \text{N}$$

This means that this force acts toward the left because the flow/pressure is pushing toward the right. Now solve for the y-component of force:

$$F_y = F_{by} + F_{sy} = \frac{\partial}{\partial t}\int_V v\rho dV + \int_{\text{area}} v\rho\vec{v}\cdot d\vec{A}$$

$$F_{by} = 0, v_1 = 0$$

$$F_{sy} = p_{outflow}A_{outflow} + p_{atm}A_2 - p_{\text{atm}}(A_{\text{outflow}} + A_2) + F_y = A_{\text{outflow}}(p_{\text{outflow}} - p_{\text{atm}}) + F_y$$

$$A_{\text{outflow}}(p_{\text{outflow}} - p_{\text{atm}}) + F_y = \int_{\text{area}} v\rho\vec{v}\cdot d\vec{A} = v_2 * (\rho V_{\text{outflow}}A_{\text{outflow}})$$

Note that the v_2-velocity component is negative (this will be accounted for later) and the flux term is positive because the area and the velocity vectors act in the same direction.

$$F_y = v_2 * (\rho v_{\text{outflow}}A_{\text{outflow}}) - A_{\text{outflow}}(p_{\text{outflow}} - p_{\text{atm}}) = v_2 * (\rho v_{\text{outflow}}A_{\text{outflow}}) - A_{\text{outflow}}(p_{\text{outflow}-\text{gauge}})$$

$$F_y = (-82.27\,\text{cm/s})(1050\ \text{kg/m}^3)(82.27\,\text{cm/s})\left(\pi\left(\frac{16\,\text{mm}}{2}\right)^2\right) - \left(\pi\left(\frac{16\,\text{mm}}{2}\right)^2\right)(85\,\text{mmHg}) = -2.42\ \text{N}$$

This means that this force acts downward. The overall force is 4.25 N acting at an angle of 214° from the positive x-axis.

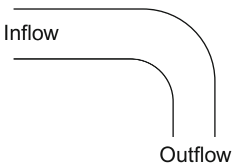

FIGURE 3.13 Brachial artery schematic for example problem.

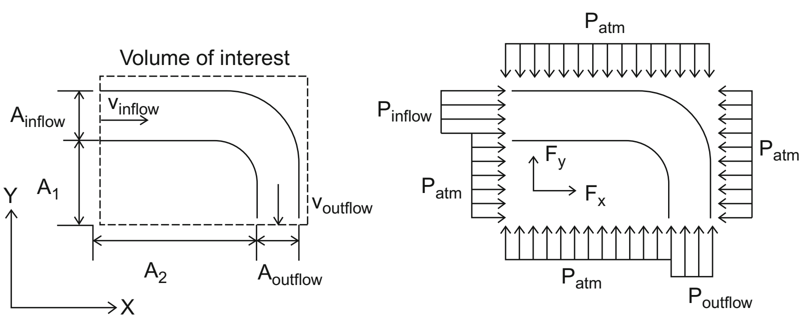

FIGURE 3.14 Free body diagram for the preceding example problem.

3.5 MOMENTUM EQUATION WITH ACCELERATION

In deriving Equation 3.33, we made an assumption that the volume of interest (and the fluid within the volume) had no acceleration at the instant in time that we were evaluating the system. Therefore, this equation does not hold for a volume of interest or system of interest that is accelerating. This is the case because typically when evaluating an accelerating system, you would use an inertial or fixed coordinate system (normally denoted as XYZ) instead of a reference (or relative) coordinate system, which follows the moving volume (normally denoted as xyz). In the previous derivations, a relative coordinate system was used for simplicity. Furthermore, Equation 3.33 does not hold for an inertial reference frame because the relative momentum for each system is not the same:

$$\vec{F} = \frac{d\vec{P}_{xyz}}{dt} \neq \frac{d\vec{P}_{XYZ}}{dt}$$

In order to develop an equivalent formulation as Equation 3.33, for an accelerating control volume, a relationship between the inertial momentum ($\vec{P}_{XYZ}$) and the control volume momentum ($\vec{P}_{xyz}$) must be found. To start, let us define Newton's second law in terms of momentum and a system of interest:

$$\vec{F} = \frac{d\vec{P}_{XYZ}}{dt} = \frac{d}{dt}\int_{\text{system}-\text{mass}} \vec{v}_{XYZ}dm = \int_{\text{system}-\text{mass}} \frac{d\vec{v}_{XYZ}}{dt}dm \tag{3.35}$$

To define the inertial velocity component in terms of the system velocity, use the following relationship:

$$\vec{v}_{XYZ} = \vec{v}_{xyz} + \vec{v}_r \tag{3.36}$$

where $\vec{v}_r$ is the velocity of the volume of interests reference frame relative to the inertial reference frame. Making the assumption that the fluid is irrotational,

$$\frac{d\vec{v}_{XYZ}}{dt} = \vec{a}_{XYZ} = \frac{d\vec{v}_{xyz}}{dt} + \frac{d\vec{v}_r}{dt} = \vec{a}_{xyz} + \vec{a}_r \tag{3.37}$$

In Equation 3.37, the first acceleration term ($\vec{a}_{XYZ}$) is the acceleration of the system relative to the inertial frame, the second acceleration term ($\vec{a}_{xyz}$) is the acceleration of the system relative to the system reference frame, and the third acceleration term ($\vec{a}_r$) is the acceleration of the system reference frame relative to the inertial reference frame. A rotational system would have multiple accelerations terms (see discussion below). Substituting the acceleration terms into Equation 3.35,

$$\vec{F} = \int_{\text{system}-\text{mass}} \frac{d\vec{v}_{XYZ}}{dt}dm = \int_{\text{system}-\text{mass}} \frac{d\vec{v}_{xyz}}{dt}dm + \int_{\text{system}-\text{mass}} \vec{a}_r dm$$

$$\vec{F} - \int_{\text{system}-\text{mass}} \vec{a}_r dm = \int_{\text{system}-\text{mass}} \frac{d\vec{v}_{xyz}}{dt}dm = \frac{d\vec{P}_{xyz}}{dt} \tag{3.38}$$

Substituting Equation 3.33 and converting the mass integral into a volume integral, Equation 3.38 becomes

$$\vec{F}_b + \vec{F}_s - \int_V \vec{a}_r \rho dV = \frac{\partial}{\partial t}\int_V \vec{v}_{xyz}\rho dV + \int_{\text{area}} \vec{v}_{xyz}\rho\vec{v}_{xyz}\cdot d\vec{A} \tag{3.39}$$

To account for the acceleration of an inertial body, relative to the inertial reference frame, the conservation of linear momentum formulation requires one extra term. When the system is not accelerating relative to the inertial frame, $\vec{a}_r$ is zero and Equation 3.39 simplifies to Equation 3.33. To apply Equation 3.39 to a system, it is required that there are two coordinate systems defined at the beginning of the problem; one is inertial coordinate system (*XYZ*), and the other stays with the moving control volume (*xyz*). This formula is valid for one instant in time, similar to Equation 3.33. However, it is possible in particular situations that the mass (i.e.,$\vec{F}_b, \rho$) and the acceleration ($\vec{a}_r$) are functions of time. This equation can easily be adapted for that scenario. Also, Equation 3.39 is a vector equation, with all velocity components related to the non-inertial reference frame (*xyz*). Written in component form, Equation 3.39 becomes

$$\begin{aligned}
\vec{F}_{bx} + \vec{F}_{sx} - \int_V \vec{a}_{rx}\rho dV &= \frac{\partial}{\partial t}\int_V u_{xyz}\rho dV + \int_{\text{area}} u_{xyz}\rho\vec{v}_{xyz}\cdot d\vec{A}\\
\vec{F}_{by} + \vec{F}_{sy} - \int_V \vec{a}_{ry}\rho dV &= \frac{\partial}{\partial t}\int_V v_{xyz}\rho dV + \int_{\text{area}} v_{xyz}\rho\vec{v}_{xyz}\cdot d\vec{A}\\
\vec{F}_{bz} + \vec{F}_{sz} - \int_V \vec{a}_{rz}\rho dV &= \frac{\partial}{\partial t}\int_V w_{xyz}\rho dV + \int_{\text{area}} w_{xyz}\rho\vec{v}_{xyz}\cdot d\vec{A}
\end{aligned} \tag{3.40}$$

Example

One of the first implantable mechanical heart valves was designed as a ball within a cage that acted as a check valve. Using the conservation of momentum (with acceleration), we will model the acceleration of the ball after it is hit by a jet of blood being ejected from the heart (Figure 3.15). The ball has a turning angle of 45° and a mass of 25 g. Blood is ejected from the heart at a velocity of 150 cm/s, through an opening with a diameter of 27 mm. Determine the velocity of the ball at 0.5 sec. Neglect any resistance to motion (except mass).

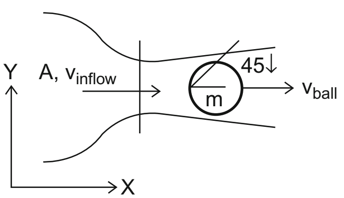

FIGURE 3.15 Acceleration of a ball and cage mechanical heart valve for the in-text problem.

Solution

The inertial reference frame is chosen at the aortic valve and the non-inertial reference frame is chosen to coincide with the ball. With the volume of interest chosen to coincide with the flow direction around the ball, Figure 3.16 represents the flow situation. We will only analyze half of the equation, because the flow is symmetrical around the uniform ball.

$$\vec{F}_{bx} + \vec{F}_{sx} - \int_V \vec{a}_{rx}\rho dV = \frac{\partial}{\partial t}\int_V u_{xyz}\rho dV + \int_{\text{area}} u_{xyz}\rho\vec{v}_{xyz}\cdot d\vec{A}$$

If we make the assumption that the blood flow is steady and uniform, the equation reduces to

$$-\int_V \vec{a}_{rx}\rho dV = \int_{\text{area}} u_{xyz}\rho\vec{v}_{xyz}\cdot d\vec{A}$$

because there are no external forces acting on the system. Substituting known values into this equation, we get

$$-\int_V \vec{a}_{rx}\rho dV = u_1\left(-\frac{\rho v_1 A_1}{2}\right) + u_2(\rho v_2 A_2)$$

We will make the assumption that as blood flows around the ball, there is no loss of velocity (and the area does not change) due to friction between the ball and blood. We will also assume that there is no change in velocity between the aortic valve location and the ball. This makes the magnitude of the following quantities

$$\frac{v_1}{2} = u_1 = u_2 = v_1 = v_2 = v_{\text{inlet}} - v_{\text{ball}}$$

which is the relative velocity, the same. Furthermore, the area can be defined as

$$\frac{A_1}{2} = A_2 = A_3 = A$$

Simplifying each term of the momentum equation and only considering the top half of the ball:

$$\int_V \vec{a}_{rx}\rho dV = \vec{a}_{rx}\rho V = \vec{a}_{rx}m = \frac{dv_{\text{ball}}}{dt}m$$

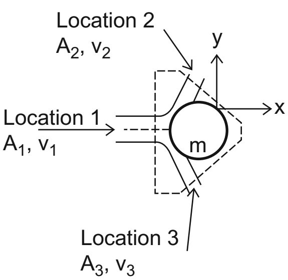

FIGURE 3.16 Free body diagram for preceding example problem.

$$u_1(-\rho v_1 A) = (v_{\text{inlet}} - v_{\text{ball}})(-\rho(v_{\text{inlet}} - v_{\text{ball}})A) = -\rho A(v_{\text{inlet}} - v_{\text{ball}})^2$$

$$u_2(\rho v_2 A) = (v_{\text{inlet}} - v_{\text{ball}})\cos(45^\circ)(\rho(v_{\text{inlet}} - v_{\text{ball}})A) = \rho A(v_{\text{inlet}} - v_{\text{ball}})^2 cos(45^\circ)$$

Substituting these values into the simplified conservation of momentum equation,

$$-\frac{dv_{\text{ball}}}{dt}m = -\rho A(v_{\text{inlet}} - v_{\text{ball}})^2 + \rho A(v_{\text{inlet}} - v_{\text{ball}})^2\cos(45^\circ) = (\cos(45^\circ) - 1)\rho A(v_{\text{inlet}} - v_{\text{ball}})^2$$

To solve this differential equation, we must separate the variables as follows:

$$\frac{dv_{\text{ball}}}{(v_{\text{inlet}} - v_{\text{ball}})^2} = \frac{(1-\cos(45^\circ))\rho A}{m}dt$$

Integrate this equation as shown:

$$\int_0^{v_{\text{ball}-\max}} \frac{dv_{\text{ball}}}{(v_{\text{inlet}} - v_{\text{ball}})^2} = \int_0^t \frac{(1-\cos(45^\circ))\rho A}{m}dt$$

$$\frac{1}{(v_{\text{inlet}} - v_{\text{ball}})}\Bigg|_0^{v_{\text{ball}-\max}} = \frac{(1-\cos(45^\circ))\rho A}{m}\Bigg|_0^t = \frac{(1-cos(45^\circ))\rho At}{m}$$

$$\frac{1}{(v_{\text{inlet}} - v_{\text{ball}})} - \frac{1}{v_{\text{inlet}}} = \frac{v_{\text{ball}}}{v_{\text{inlet}}(v_{\text{inlet}} - v_{\text{ball}})} = \frac{(1-\cos(45^\circ))\rho At}{m}$$

Solving this equation for v_{ball},

$$\frac{v_{\text{ball}}}{v_{\text{inlet}}^2 - v_{\text{inlet}}v_{\text{ball}}} = \frac{(1-cos(45^\circ))\rho At}{m}$$

$$v_{\text{ball}} = (v_{\text{inlet}}^2 - v_{\text{inlet}}v_{\text{ball}})\left(\frac{(1-cos(45^\circ))\rho At}{m}\right)$$

$$= v_{\text{inlet}}^2\left(\frac{(1-\cos(45^\circ))\rho At}{m}\right) - v_{\text{inlet}}v_{\text{ball}}\left(\frac{(1-\cos(45^\circ))\rho At}{m}\right)$$

$$v_{\text{ball}} + v_{\text{inlet}}v_{\text{ball}}\left(\frac{(1-\cos(45^\circ))\rho At}{m}\right) = v_{\text{ball}}\left(1 + v_{\text{inlet}}\left(\frac{(1-\cos(45^\circ))\rho At}{m}\right)\right)$$

$$= v_{\text{inlet}}^2\left(\frac{(1-\cos(45^\circ))\rho At}{m}\right)$$

$$v_{\text{ball}} = v_{\text{inlet}}^2\left(\frac{(1-\cos(45^\circ))\rho At}{m\left(1 + v_{\text{inlet}}\left(\frac{(1-\cos(45^\circ))\rho At}{m}\right)\right)}\right) = v_{\text{inlet}}^2\left(\frac{(1-\cos(45^\circ))\rho At}{m + v_{\text{inlet}}((1-\cos(45^\circ))\rho At)}\right)$$

$$= (150\ \text{cm/s})^2\left(\frac{(1-\cos(45^\circ))(1050\ \text{kg/m}^3)\frac{\pi}{2}\left(\frac{27\ \text{mm}}{2}\right)^2 t}{25\text{g} + (150\ \text{cm/s})\left((1-\cos(45^\circ))(1050\ \text{kg/m}^3)\frac{\pi}{2}\left(\frac{27\ \text{mm}}{2}\right)^2 t\right)}\right)$$

$$= \frac{198\ \ \text{gm/s}^2 * t}{25\ \ \text{g} + 132\ \text{g/s} * t}$$

To account for the bottom half of the flow,

$$v_{\text{ball}} = 2\left(\frac{198 \ \text{g}*\text{m/s}^2 * t}{25 \ \text{g} + 132 \ \text{g/s} * t}\right)$$

At t = 0.5 sec,

$$v_{\text{ball}} = 2\left(\frac{198 \ \text{g}*\text{m/s}^2 * 0.5 \ \text{s}}{25 \ \text{g} + 132 \ \text{g/s} * 0.5 \ \text{s}}\right) = 218 \ \text{cm/s}$$

This is consistent with a rapid opening of the valve, but the total length that the ball would traverse would only be approximately 4 cm (at most). Over time, the velocity of the ball would follow a logarithmic relationship (Figure 3.17), if there was no mechanism to stop the ball from moving (i.e., the cage).

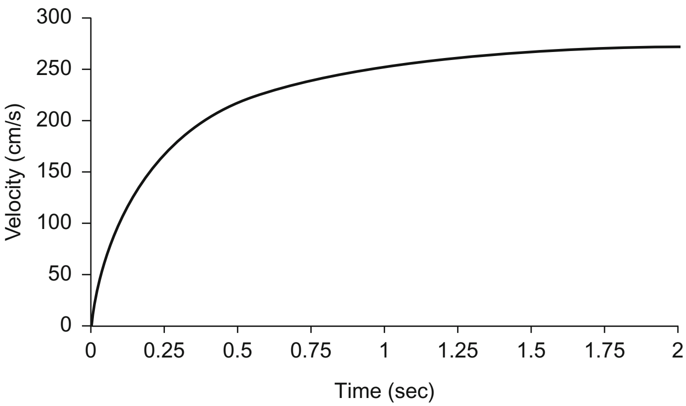

FIGURE 3.17 Velocity of the ball with respect to time.

We made the assumption in the derivation of Equation 3.40, that the flow was irrotational, and therefore it only experienced pure translation. We will not show the derivation of the formula here, but the most general formula for the conservation of momentum must include all possible velocity components. This formula takes the form of

$$\begin{aligned} &\vec{F}_b + \vec{F}_s - \int_V (\vec{a}_r + 2\vec{\omega} \times \vec{v}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{r})\rho dV \\ &= \frac{\partial}{\partial t}\int_V \vec{v}_{xyz}\rho dV + \int_{\text{area}} \vec{v}_{xyz}\rho\vec{v}_{xyz} \cdot d\vec{A} \end{aligned} \tag{3.42}$$

where ω is the angular velocity (note: $\vec{\omega} \times \vec{v}_{xyz}$ is the Coriolis acceleration, $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is the centripetal acceleration, and $\vec{\omega} \times \vec{r}$ is the tangential acceleration due to angular velocity). This formula would be used if the fluid elements rotate and translate about each other or some reference coordinate axis (XYZ).

3.6 THE FIRST AND SECOND LAWS OF THERMODYNAMICS

The conservation of energy within a system is defined by the first law of thermodynamics, which is

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

$\dot{Q}$ is the time rate of change of heat transfer and is positive when heat is added to the system. $\dot{W}$ is the time rate of change of work and is positive when work is done by the system. The energy of a system can be defined as

$$E = \int_V e\rho dV = \int_V \left(u + \frac{v^2}{2} + gz\right)\rho dV \tag{3.43}$$

where e is the energy per unit mass, u is the specific internal energy of the system, v is the speed of the system (not velocity), and z is the height of the system relative to a reference point. From physics class, this is similar to a statement of the total energy of the system, including potential energy, kinetic energy, and any other internal energy. In developing Equation 3.22, we stated that for energy conservation the system property was E and the inherent property was e. Substituting these values into Equation 3.22, we have a statement for the conservation of energy:

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\int_V e\rho dV + \int_{\text{area}} e\rho\vec{v}\cdot d\vec{A} \tag{3.44}$$

If we define the system to be the same as the volume of interest at the instant in time that is of interest to us, we can make the following statement:

$$\dot{Q} - \dot{W}\Big|_{\text{system}} = \dot{Q} - \dot{W}\Big|_{\text{volume of interest}}$$

Substituting this into Equation 3.44, we get

$$\begin{aligned}\dot{Q} - \dot{W} &= \frac{\partial}{\partial t}\int_V e\rho dV + \int_{\text{area}} e\rho\vec{v}\cdot\left|d\vec{A}\right| \\ &= \frac{\partial}{\partial t}\int_V \left(u + \frac{v^2}{2} + gz\right)\rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz\right)\rho\vec{v}\cdot d\vec{A}\end{aligned} \tag{3.45}$$

In general, the rate of work is hard to quantify in fluid mechanics. Typically, in fluid mechanics, work is divided into four categories: work from normal stresses (W_n), work from shear stresses (W_{sh}), shaft work (W_s), and any other work (W_o).

From a physics course, you should remember that work is defined by force multiplied by the distance that the force acts over. When describing the work on a differential element,

$$dW = \vec{F}\cdot d\vec{x}$$

The normal force, acting on a differential element would be defined as

$$d\vec{F} = \sigma_N d\vec{A}$$

To define the time rate of change of work,

$$\dot{W}_n = \lim_{\Delta t \to 0} \frac{\partial W}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F} \cdot d\vec{x}}{\Delta t} = \sigma_N d\vec{A} \cdot \vec{v}$$

because $\frac{d\vec{x}}{dt} = \vec{v}$. Therefore, the total work done by normal forces is

$$\dot{W}_n = -\int_{\text{area}} \sigma_N \vec{v} \cdot d\vec{A} \tag{3.46}$$

where the negative sign is needed to quantify the work done on the control volume instead of by the control volume.

The work of a shear force is defined in a similar way. Remember that shear force is defined as

$$d\vec{F} = \vec{\tau}\, dA$$

In this formulation, the shear stress is the vector quantity (to provide directionality of the stress which is different from the area vectors direction) not the area normal vector. Using the same process as above, the work of shear becomes

$$\dot{W}_{sh} = -\int_{\text{area}} \vec{\tau} \cdot \vec{v}\, dA \tag{3.47}$$

The work done by a shaft is not applicable to many biofluid mechanics problems, but would be defined as the negative of work input into the shaft to move the fluid. Similarly, other work would need to be defined by the type of work that is being done. For instance, if energy from an x-ray is being absorbed into the fluid, this can be considered as work being absorbed by the system. Using these definitions, Equation 3.45 becomes

$$\dot{Q} + \int_{\text{area}} \sigma_N \vec{v} \cdot d\vec{A} + \int_{\text{area}} \vec{\tau} \cdot \vec{v}\, dA - \dot{W}_{\text{shaft}} - \dot{W}_{\text{other}}$$
$$= \frac{\partial}{\partial t} \int_V \left(u + \frac{v^2}{2} + gz \right) \rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz \right) \rho \vec{v} \cdot d\vec{A}$$

$$\dot{Q} + \int_{\text{area}} \vec{\tau} \cdot \vec{v}\, dA - \dot{W}_{\text{shaft}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t} \int_V \left(u + \frac{v^2}{2} + gz \right) \rho dV$$
$$+ \int_{\text{area}} \left(u + \frac{v^2}{2} + gz \right) \rho \vec{v} \cdot d\vec{A} - \int_{\text{area}} \sigma_N \vec{v} \cdot d\vec{A}$$

We make use of the definition of specific volume (ν):

$$\rho = \frac{1}{\nu}$$

and then we combine like integrals to get

$$\dot{Q} + \int_{\text{area}} \vec{\tau} \cdot \vec{v}\, dA - \dot{W}_{\text{shaft}} - \dot{W}_{\text{other}}$$

$$= \frac{\partial}{\partial t} \int_V \left(u + \frac{v^2}{2} + gz\right) \rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz\right) \rho \vec{V} \cdot d\vec{A} - \int_{\text{area}} \sigma_N \nu \rho \vec{v} \cdot d\vec{A}$$

$$= \frac{\partial}{\partial t} \int_V \left(u + \frac{v^2}{2} + gz\right) \rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz - \sigma_N \nu\right) \rho \vec{v} \cdot d\vec{A}$$

From a previous discussion, we have defined that the normal stress is equal to the negative of the hydrostatic pressure (in most cases without large viscous effects), therefore,

$$\dot{Q} + \int_{\text{area}} \vec{\tau} \cdot \vec{v}\, dA - \dot{W}_{\text{shaft}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t} \int_V \left(u + \frac{v^2}{2} + gz\right) \rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz + p\nu\right) \rho \vec{v} \cdot d\vec{A} \tag{3.48}$$

Example

One of the functions of the cardiovascular system is to act as a heat exchanger, to maintain body temperature (see Figure 3.18). Calculate the rate of heat transfer through a capillary bed, assuming that the blood velocity into the capillary is 100 mm/s and the flow velocity out of the capillary bed is 40 mm/s. The pressure on the arterial side is 20 mmHg, and the pressure on the venous side is 12 mmHg. Assume that the arteriole diameter is 75 μm and the venule diameter is 50 μm. The temperature on the arterial side is 35°C, and the temperature on the venous side is 33°C. Assume that the power put into the system throughout the muscular system is 15 μW.

Solution

To solve this problem, we will make the assumptions that the flow is steady, uniform, the height difference between the arterial side and venous side is zero, and there is negligible internal energy or work done by stresses in the capillary bed. The conservation of mass does not hold across the capillary bed, because fluid is lost into the interstitial space:

$$\dot{Q} + \int_{\text{area}} \vec{\tau} \cdot \vec{v}\, dA - \dot{W}_{\text{shaft}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t} \int_V \left(u + \frac{v^2}{2} + gz\right) \rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz + p\nu\right) \rho \vec{v} \cdot d\vec{A}$$

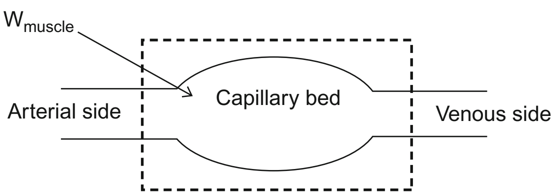

FIGURE 3.18 Schematic of a capillary heat exchanger for the example problem.

$$\dot{Q} = \dot{W}_{\text{muscle}} + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz + p\nu\right)\rho\vec{v}\cdot d\vec{A}$$

$$\dot{Q} = \dot{W}_{\text{muscle}} + \left(\frac{v_1^2}{2} + gz_1 + p_1\nu_1\right)(-\rho v_1 A_1) + \left(\frac{v_2^2}{2} + gz_2 + p_2\nu_2\right)(\rho v_2 A_2)$$

Substituting known values into this equation,

$$\begin{aligned}\dot{Q} &= 15\ \text{W} + \left(\frac{(100\ \text{mm/s})^2}{2} + gz_1 + \frac{20\ \text{mmHg}}{1050\ \text{kg/m}^3}\right)\left(-(1050\ \text{kg/m}^3)(100\ \text{mm/}s)\pi\left(\frac{75\ \mu\text{m}}{2}\right)^2\right) \\ &\quad + \left(\frac{(40\ \text{mm/s})^2}{2} + gz_2 + \frac{12\ \text{mmHg}}{1050\ \text{kg/m}^3}\right)\left((1050\ \text{kg/m}^3)(40\ \text{mm/s})\pi\left(\frac{50\ \mu\text{m}}{2}\right)^2\right) \\ &= 15\ \text{W} + (-1\ \mu W - (4.64_E - 4\ \text{g/s})\text{g}z_1) + (0.126\ \mu W + (8.25_E - 5\ \text{g/s})gz_2) \\ &= 14.126\ \text{W} - (3.814_E - 4\ \text{g/s})g(z_2 - z_1)\end{aligned}$$

We are making the assumption that there is no height difference between the arterial side and venous side ($z_2 = z_1$).

Therefore, the rate of heat transfer is 14.126 μW. For every millimeter difference in height, the rate of heat transfer would change by approximately 4 nW. The addition or subtraction of heat would depend on if the arterial side or the venous side was higher.

Example

Calculate the time rate of change of mass flow rate (ρvA) of air entering the lungs. Assume that the lungs have a capacity of 6 L. The temperature of the lungs is 37°C. The air pressure inside of the lungs is 0.98 atm. At the instant that air enters the lungs, the temperature of the lungs raises by 0.0001°C/s. The height of the trachea is 20 cm. Assume that there is no work added to the system. Assume that air behaves as an ideal gas. Assume that the velocity is slow within the trachea.

Solution

$$\begin{aligned}\dot{Q} + \int_{\text{area}} \vec{\tau}\cdot\vec{v}\,dA - \dot{W}_{\text{shaft}} - \dot{W}_{\text{other}} &= \frac{\partial}{\partial t}\int_V \left(u + \frac{v^2}{2} + gz\right)\rho dV \\ &\quad + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz + p\nu\right)\rho\vec{v}\cdot d\vec{A}\end{aligned}$$

From the assumptions made, the given equation can simplify to

$$0 = \frac{\partial}{\partial t}\int_V (u + gz)\rho dV + \int_{\text{area}} (u + gz + p\nu)\rho\vec{v}\cdot d\vec{A}$$

$$0 = \frac{\partial}{\partial t}\int_V (u+gz)\rho dV + (u+gz+p\nu)(-\rho v A) = \frac{\partial}{\partial t}\left[(u+gz)m\right] - (u+gz+p\nu)(\dot{m})$$

$$= \frac{\partial}{\partial t}(um+gzm) - (u+gz+p\nu)(\dot{m})$$

$$= \frac{\partial}{\partial t}(um) + \frac{\partial}{\partial t}(gzm) - u\frac{\partial m}{\partial t} - gz\frac{\partial m}{\partial t} - p\nu\frac{\partial m}{\partial t}$$

$$= m\frac{\partial u}{\partial t} + u\frac{\partial m}{\partial t} + m\frac{\partial gz}{\partial t} + gz\frac{\partial m}{\partial t} - u\frac{\partial m}{\partial t} - gz\frac{\partial m}{\partial t} - p\nu\frac{\partial m}{\partial t} = m\frac{\partial u}{\partial t} - p\nu\frac{\partial m}{\partial t}$$

$$\dot{m} = \frac{m}{p\nu}\frac{\partial u}{\partial t}$$

Because we are making the assumption that air will behave like an ideal gas, we can make the following substitutions:

$$m = \rho V$$

$$p\nu = RT$$

$$\frac{\partial u}{\partial t} = C_v\frac{dT}{dt}$$

to get

$$\dot{m} = \frac{m}{p\nu}\frac{\partial u}{\partial t} = \frac{\rho V}{RT}C_v\frac{dT}{dt} = \frac{pV}{R^2T^2}C_v\frac{dT}{dt}$$

$$\dot{m} = \frac{3.98\ \text{atm} * 6\text{L}}{(287\ \text{Nm/kgk})^2((37+273)\text{K})^2}(717\ \text{Nm/kgk})(0.0001\text{K/s}) = 3.24_E\,{-4}\ \text{g/min}$$

This is consistent with normal breathing.

The second law of thermodynamics is a statement about the disorder of a system. It states that the change of entropy of a system is greater than or equal to the amount of heat added to the system at a particular temperature:

$$dS \geq \frac{dQ}{T} \tag{3.49}$$

The time rate of change of entropy can therefore be defined as

$$\frac{dS}{dt} \geq \frac{\dot{Q}}{T} \tag{3.50}$$

for one specific volume of interest. When developing Equation 3.22, we stated that for energy conservation the system property was S and the inherent property was s (entropy per unit mass). Substituting these values into Equation 3.22, we have a statement for the conservation of energy related to the second law of thermodynamics:

$$\frac{dS}{dt} = \frac{\partial}{\partial t}\int_V s\rho dV + \int_{\text{area}} s\rho\vec{v}\cdot d\vec{A} \tag{3.51}$$

Substituting Equation 3.50 into 3.51,

$$\frac{\partial}{\partial t}\int_V s\rho dV + \int_{\text{area}} s\rho\vec{v}\cdot d\vec{A} \geq \frac{\dot{Q}}{T} \tag{3.52}$$

Equation 3.52 is a statement of the second law of thermodynamics, which can be directly applied to fluid mechanics problems. In some instances, it may be useful to know that

$$\left.\frac{\dot{Q}}{T}\right)_V = \int_{\text{area}} \frac{\dot{Q}}{AT} dA \tag{3.53}$$

$\frac{\dot{Q}}{A}$ is the heat flux along one particular area. This is normally constant for one particular surface area of interest.

3.7 THE NAVIER-STOKES EQUATIONS

In the previous sections, we have applied various physical laws to a fluid volume of interest. However, to obtain an equation that describes the fluid motion at any time or location within the flow field, it is easier to apply Newton's second law of motion to a particle. For a system such as this, Newton's law becomes

$$d\vec{F} = dm\frac{d\vec{v}}{dt} \tag{3.54}$$

The derivation for the acceleration of a fluid particle has already been shown in Chapter 2. Using that relationship for particle acceleration, Newton's law becomes

$$d\vec{F} = dm\left(\frac{\partial\vec{v}}{\partial t} + u\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + w\frac{\partial\vec{v}}{\partial z}\right) \tag{3.55}$$

As discussed before, the forces on a fluid particle can be body forces or surface forces. To define these forces, let us look at the forces that act on a differential element with mass dm and volume $dV = dxdydz$. As done previously to describe the pressure acting on a differential element, assume that the stresses acting at the cubes center (denoted as p) are ω_{xx}, τ_{yx}, and τ_{zx} (Figure 3.19). Note that all of these stresses act in the x-direction on this figure.

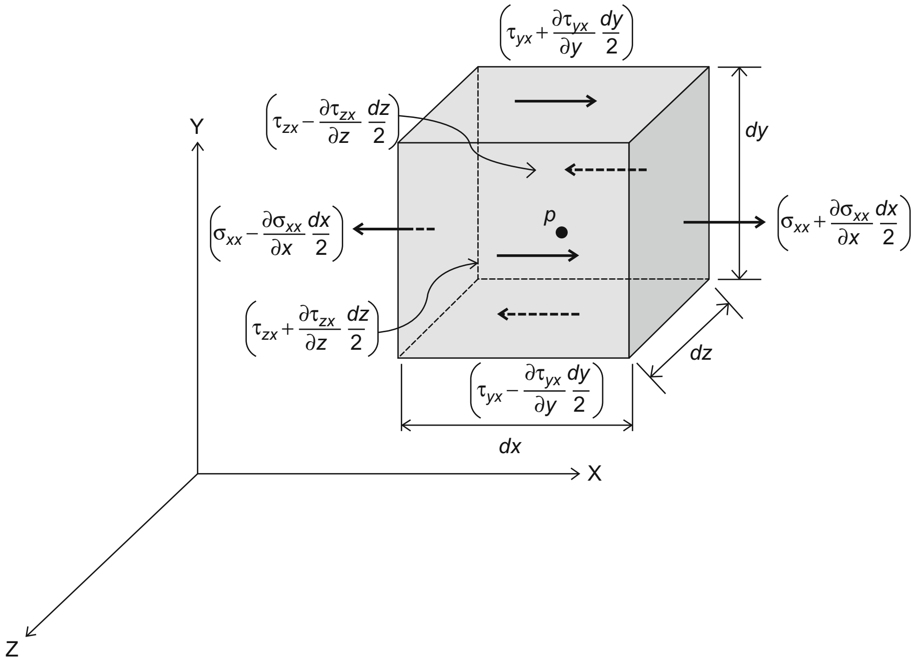

FIGURE 3.19 The normal and shear stresses acting in the x-direction on a differential fluid element. The stresses that act in the other Cartesian directions can be derived in a similar manner. Recall that only six of these stress values are independent for momentum conservation.

Quantifying the stresses in the x-direction as a component of the total surface forces,

$$\begin{aligned} dF_{sx} = & \left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz-\left(\sigma_{xx}-\frac{\partial\sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz \\ & -\left(\tau_{yx}-\frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dxdz+\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy-\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dxdy \\ = & \left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)dxdydz \end{aligned} \tag{3.56}$$

Using a similar analysis for each of the remaining two directions and assuming that the gravitational force is the only body force, the total force in each direction becomes

$$\begin{aligned} dF_x = dF_{bx}+dF_{sx} &= \left(\rho g_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)dxdydz \\ dF_y = dF_{by}+dF_{sy} &= \left(\rho g_y+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)dxdydz \\ dF_z = dF_{bz}+dF_{sz} &= \left(\rho g_z+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)dxdydz \end{aligned} \tag{3.57}$$

For this formulation, gravity does not need to align with a particular Cartesian direction. Substituting Equation 3.57 into Equation 3.55 and writing it in terms of vector components gives us

$$\left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz = dm\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$
$$\left(\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) dxdydz = dm\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$
$$\left(\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) dxdydz = dm\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Using the relationship that $dm = \rho dV = \rho dxdydz$, the equations of motion become

$$\begin{aligned}
\left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) &= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho\frac{Du}{Dt} \\
\left(\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) &= \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho\frac{Dv}{Dt} \\
\left(\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) &= \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho\frac{Dw}{Dt}
\end{aligned} \tag{3.58}$$

Equation 3.58 is the differential equations of motion, which are valid for any fluid that is a continuum and for any fluid that has the force of gravity as the only body force. In Chapter 2, we defined the normal stress as a function of hydrostatic pressure and the shear stresses as a function of viscosity and shear rate (which is a function of velocity). The following definitions apply:

$$\begin{aligned}
\sigma_{xx} &= -p - \frac{2}{3}\mu\nabla\cdot\vec{v} + 2\mu\frac{\partial u}{\partial x} \\
\sigma_{yy} &= -p - \frac{2}{3}\mu\nabla\cdot\vec{v} + 2\mu\frac{\partial v}{\partial x} \\
\sigma_{zz} &= -p - \frac{2}{3}\mu\nabla\cdot\vec{v} + 2\mu\frac{\partial w}{\partial x} \\
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
\tau_{xz} &= \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\
\tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned} \tag{3.59}$$

where ∇ is the gradient operator and is defined as

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

where f is any function in the Cartesian coordinate system (note that the gradient function can be calculated in any coordinate system for any function; we are just highlighting the Cartesian coordinate system for this analysis).

Substituting Equations 3.59 into Equations 3.58, the equations of motion become

$$
\begin{aligned}
\rho\frac{Du}{Dt} &= \left(\rho g_x + \frac{\partial\left(-p-\frac{2}{3}\mu\nabla\cdot\vec{v}+2\mu\frac{\partial u}{\partial x}\right)}{\partial x} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)}{\partial y} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)}{\partial z}\right) \\
&= \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\mu\nabla\cdot\vec{v}\right)\right]}{\partial x} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)}{\partial y} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)}{\partial z} \\
\rho\frac{Dv}{Dt} &= \left(\rho g_y + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)}{\partial x} + \frac{\partial\left(-p-\frac{2}{3}\mu\nabla\cdot\vec{v}+2\mu\frac{\partial v}{\partial x}\right)}{\partial y} + \frac{\partial\left(\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)}{\partial z}\right) \\
&= \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial\left[\mu\left(2\frac{\partial v}{\partial x}-\frac{2}{3}\mu\nabla\cdot\vec{v}\right)\right]}{\partial y} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)}{\partial x} + \frac{\partial\left(\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)}{\partial z} \\
\rho\frac{Dw}{Dt} &= \left(\rho g_z + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)}{\partial x} + \frac{\partial\left(\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)}{\partial y} + \frac{\partial\left(-p-\frac{2}{3}\mu\nabla\cdot\vec{v}+2\mu\frac{\partial w}{\partial x}\right)}{\partial z}\right) \\
&= \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial\left[\mu\left(2\frac{\partial w}{\partial x}-\frac{2}{3}\mu\nabla\cdot\vec{v}\right)\right]}{\partial z} + \frac{\partial\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)}{\partial x} + \frac{\partial\left(\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)}{\partial y}
\end{aligned}
\tag{3.60}
$$

Equations 3.60 are the full Navier-Stokes equations that are valid for any fluid. If we assume that the fluid is incompressible and the viscosity is uniform and constant, the equations simplify to

$$
\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) &= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) &= \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \\
\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) &= \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}
\tag{3.61}
$$

Equation 3.61 is the form of the Navier-Stokes equations that will be used often in this textbook. In many biofluid mechanics examples, it is however more useful to solve the Navier-Stokes equations in a cylindrical coordinate system. The Navier-Stokes equations in cylindrical coordinates are as follows for incompressible fluids with a constant viscosity:

$$
\begin{aligned}
&\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right] \\
&\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \\
&\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right) = \rho g_\theta - \frac{\partial p}{\partial\theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]
\end{aligned}
\tag{3.62}
$$

The following two examples illustrate how to use either the Cartesian form or the cylindrical form of the Navier-Stokes equations. Remember that the usefulness of these equations is that the fully developed velocity profile of any flowing fluid can be determined. If the flow is one-dimensional, then this can be easily solved by hand.

Example

Find an expression for the velocity profile and the shear stress (τ_{xy}) distribution for blood flowing in an arteriole with a diameter of 500 μm. Use the Navier-Stokes equations for Cartesian coordinates to solve this problem. The pressure driving this flow is given in Figure 3.20.

Solution

To solve this problem, assume that $v_y = v_z = 0$ and v_x is a function of y only. Assume that the viscosity is constant and that the flow is incompressible and steady.

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = -\frac{\partial p}{\partial x} + \mu\frac{d^2 u}{dy^2}$$

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \rightarrow d^2 u = \frac{1}{\mu}\frac{\partial p}{\partial x}dy^2$$

$$\int d^2 u = \int \frac{1}{\mu}\frac{\partial p}{\partial x}dy^2$$

$$du = \left(\frac{y}{\mu}\frac{\partial p}{\partial x} + c_1\right)dy$$

$$\int du = \int\left(\frac{y}{\mu}\frac{\partial p}{\partial x} + c_1\right)dy$$

$$u(y) = \frac{y^2}{2\mu}\frac{\partial p}{\partial x} + c_1 y + c_2$$

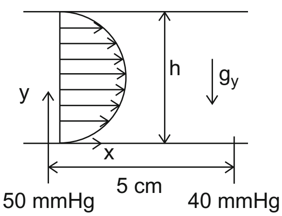

FIGURE 3.20 Pressure driven flow in an arteriole for example problem.

To find the exact solution, we need boundary conditions for the particular flow scenario. Due to the no-slip boundary condition, we know that the velocity at both walls is zero and that the shear stress at the centerline is zero. Therefore, our boundary conditions are

$$u(0) = 0$$
$$u(h) = 0$$
$$\frac{du(h/2)}{dy} = 0$$

Using two of these conditions to solve for the integration constants,

$$u(0) = \frac{0^2}{2\,\mu}\frac{\partial p}{\partial x} + c_1 0 + c_2 = 0$$
$$c_2 = 0$$
$$\frac{du(h/2)}{dy} = \frac{h/2}{\mu}\frac{\partial p}{\partial x} + c_1$$
$$c_1 = -\frac{h}{2\mu}\frac{\partial p}{\partial x}$$

Substituting the values for these integration constants into the velocity equation,

$$u(y) = \frac{y^2}{2\mu}\frac{\partial p}{\partial x} - \frac{h}{2\mu}\frac{\partial p}{\partial x}y = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - hy)$$

For this particular flow scenario,

$$\mu = 3.5\ \text{cP}$$
$$\frac{\partial p}{\partial x} = \frac{40\ \text{mmHg} - 50\ \text{mmHg}}{5\ \text{cm} - 0\ \text{cm}} = -2\ \text{mmHg/cm}$$
$$h = 500\ \mu\text{m}$$
$$u(y) = \frac{1}{2 * 3.5\ \text{cP}}(-2\ \text{mmHg/cm})(y^2 - 500\ \mu\text{m}y) = \frac{-3.81}{\mu\text{ms}}(y^2 - 500\ \mu\text{m}y)$$

The shear stress profile for this particular flow is equal to

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\frac{\partial u}{\partial y}$$
$$\tau_{xy} = \mu\left(\frac{y}{\mu}\frac{\partial p}{\partial x} - \frac{h}{2\mu}\frac{\partial p}{\partial x}\right) = \frac{\partial p}{\partial x}\left(y - \frac{h}{2}\right)$$

Substituting the appropriate known values for this particular scenario,

$$\tau_{xy} = \mu\left(\frac{y}{\mu}\frac{\partial p}{\partial x} - \frac{h}{2\mu}\frac{\partial p}{\partial x}\right) = \frac{\partial p}{\partial x}\left(y - \frac{h}{2}\right)$$
$$= -0.267\frac{\text{dyne}}{\text{cm}^2\mu\text{m}}(y - 250\ \mu\text{m})$$

Example

Find an expression for the velocity profile and the shear stress distribution for blood flowing in an arteriole with a diameter of 500 μm. Use the Navier-Stokes equations for cylindrical coordinates to solve this problem. The pressure driving this flow is given in Figure 3.21.

Solution

To solve this problem, assume that $v_r = v_\theta = 0$ and v_z is a function of r only. Assume that the viscosity is constant and that the flow is incompressible and steady.

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

$$\frac{r}{\mu}\frac{\partial p}{\partial z} = \frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

$$\int \frac{r}{\mu}\frac{\partial p}{\partial z}dr = \int d\left(r\frac{dv_z}{dr}\right)$$

$$\frac{r^2}{2\mu}\frac{\partial p}{\partial z} + c_1 = r\frac{dv_z}{dr} \rightarrow \frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{c_1}{r} = \frac{dv_z}{dr}$$

$$\int\left(\frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{c_1}{r}\right)dr = \int dv_z$$

$$v_z(r) = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + c_1 ln(r) + c_2$$

To find the exact solution, we need boundary conditions for the particular flow scenario. Due to the no-slip boundary condition, we know that the velocity at the wall is zero and that the shear stress at the centerline is zero. Therefore, our boundary conditions are

$$v_z(R) = 0$$

$$\frac{dv_z(0)}{dr} = 0$$

Note that we do not have a boundary condition of $v_z(-R) = 0$, because in cylindrical coordinates there is no negative radial direction. This location would be associated with 180° in the

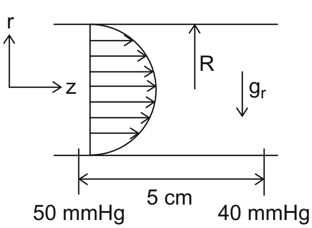

FIGURE 3.21 Pressure-driven flow in an arteriole with cylindrical coordinates for the in text example. This is the same image as Figure 3.20, but choosing a different coordinate system to illustrate the usage of Cartesian coordinates versus cylindrical coordinates.

theta direction and $+R$ in the radial direction. Using these conditions to solve for the integration constants,

$$\frac{dv_z(0)}{dr} = \frac{0}{2\mu}\frac{\partial p}{\partial z} + \frac{c_1}{0} = 0$$

Due to the discontinuity at $r = 0$, the only way for this equation to be valid is that $c_1 = 0$. Therefore, the discontinuity is removed. Using the second boundary condition,

$$v_z(R) = \frac{R^2}{4\mu}\frac{\partial p}{\partial z} + c_2 = 0$$

$$c_2 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}$$

Substituting the values for the integration constants into the velocity equation,

$$v_z(r) = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} - \frac{R^2}{4\mu}\frac{\partial p}{\partial z} = \frac{R^2}{4\mu}\frac{\partial p}{\partial z}\left(\left(\frac{r}{R}\right)^2 - 1\right)$$

For this particular flow scenario, using the same values as the previous example,

$$v_z(r) = \frac{R^2}{4\mu}\frac{\partial p}{\partial z}\left(\left(\frac{r}{R}\right)^2 - 1\right) = -11.9 \text{ cm/s}\left(\frac{r^2}{62,500 \ \mu\text{m}^2} - 1\right)$$

The shear stress distribution is

$$\tau_{zr} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) = \mu\frac{\partial v_z}{\partial r}$$

$$\tau_{zr} = \mu\left(\frac{r}{2\mu}\frac{\partial p}{\partial z}\right) = \frac{r}{2}\frac{\partial p}{\partial z} = -0.1332 \ \text{dyne/cm}^2\mu\text{m} * r$$

3.8 BERNOULLI EQUATION

The Bernoulli equation is a useful formula that relates the hydrostatic pressure, the fluid height, and the speed of a fluid element. However, there are a few important assumptions that are made to derive this formula, which makes this powerful equation not necessarily useful in many biofluid mechanics applications. Although as a back-of-the-envelope calculation, the Bernoulli equation can approximate the real flow situation reasonably well. To derive this equation, the conservation of mass and conservation of momentum equations are simplified by making the assumptions that the flow is steady, incompressible, and invisicid (has no viscosity).

To derive the Bernoulli equation, let us follow a differential volume of fluid in an expanding streamline (Figure 3.22). The fluid properties at the inlet will be denoted as p_i, v_i, A_i, and ρ. The fluid properties at the outlet will be denoted as $p_i + dp_i$, $v_i + dv_i$, $A_i + dA_i$, and ρ. This same analysis can be conducted for a reducing streamline, where the solution would include negative differential changes, as necessary.

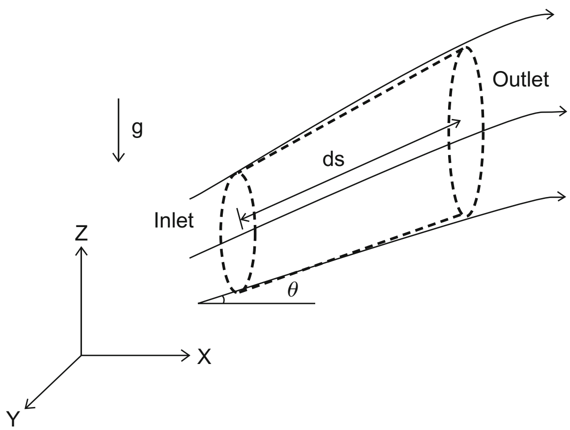

FIGURE 3.22 A differential volume of fluid following expanding streamlines (streamlines are the curved arrows in the figure). The expansion causes an increase in area at the outlet as compared to the inlet.

Applying the conservation of mass to this condition, we find that

$$0 = \frac{\partial}{\partial t}\int_V \rho dV + \int_{\text{area}} \rho \vec{v} \cdot d\vec{A}$$

$$0 = \int_{\text{area}} \rho \vec{v} \cdot d\vec{A} = -\rho v_i A_i + \rho(v_i + dv_i)(A_i + dA_i)$$

because we assumed that the flow is steady and incompressible. Simplifying the previous equation, we can obtain

$$\rho v_i A_i = \rho(v_i A_i + v_i dA_i + A_i dv_i + dv_i dA_i) = \rho v_i A_i + \rho v_i dA_i + \rho A_i dv_i + \rho dv_i dA_i \tag{3.63}$$

Remember that the product of two differentials ($dv_i dA_i$) is going to be negligible compared to the remaining terms, which allows us to simplify Equation 3.63 to

$$0 = v_i dA_i + A_i dv_i \tag{3.64}$$

Now we will simplify the conservation of momentum equation in the streamline direction (s). The conservation of momentum states that

$$F_{bs} + F_{ss} = \frac{\partial}{\partial t}\int_V v_s \rho dV + \int_{\text{area}} v_s \rho \vec{v} \cdot d\vec{A}$$

for the streamline direction. Because we are assuming steady flow, this formula simplifies to

$$F_{bs} + F_{ss} = \int_{\text{area}} v_s \rho \vec{v} \cdot d\vec{A} \tag{3.65}$$

Because the fluid is invisicid, the only forces that arise are from the pressure acting on the two surfaces and the surrounding fluid and the body force due to gravity.

These forces are

$$F_{ss} = p_i A_i - (p_i + dp_i)(A_i + dA_i) + \left(p_i + \frac{dp_i}{2}\right) dA_i = -A_i dp_i$$
$$F_{bs} = g_s \rho dV = (-g\sin\theta)\rho\left(A + \frac{dA_i}{2}\right) ds = -\rho g\left(A_i + \frac{dA_i}{2}\right) dz \quad (3.66)$$

because $\sin\theta ds = dz$. The differential volume term (dV in the F_{bs} equation) is an approximation of the volume using the midpoint area. The same analysis was used to approximate the pressure on the surrounding fluid; that is, use the midpoint pressure as the approximate pressure on the fluid. The flux term in Equation 3.65 is equal to

$$\int_{\text{area}} v_s \rho \vec{v} \cdot d\vec{A} = v_i(-\rho v_i A_i) + (v_i + dv_i)(\rho(v_i + dv_i)(A_i + dA_i))$$

making use of the inflow/outflow conditions. From the continuity equation (prior to simplifying the term),

$$v_i(-\rho v_i A_i) + (v_i + dv_i)(\rho(v_i + dv_i)(A_i + dA_i)) = v_i(-\rho v_i A_i) + (v_i + dv_i)(\rho v_i A_i) = \rho v_i A_i dv_i \quad (3.67)$$

Substituting Equations 3.66/3.67 into the momentum equation (Equation 3.65),

$$-A_i dp - \rho g\left(A_i + \frac{dA_i}{2}\right) dz = -A_i dp_i - \rho g A_i dz = \rho v_i A_i dv_i$$

If we divide this equation by ρA_i, and simplify the velocity derivative,

$$0 = v_i dv_i + \frac{dp_i}{\rho} + g dz = d\left(\frac{v_i^2}{2}\right) + \frac{dp_i}{\rho} + g dz$$

Integrating this equation and dropping the subscripts, we obtain the Bernoulli equation:

$$\frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant} \quad (3.68)$$

As we stated before, the Bernoulli equation is a powerful equation which relates the flow speed, the hydrostatic pressure, and the height to a constant. It can only be applied to a situation where the flow is steady, invisicid, and incompressible. In developing this relationship, we used a differential element, where these three criteria were valid. In most cases, it will not be easy to justify the use of the Bernoulli equation instead of the Navier-Stokes equations, the Conservation of Mass, and the Conservation of Momentum. However, as our example will show, we can use the Bernoulli equation as an approximation for various flow situations. In this simplified form, the Bernoulli equation is a statement of the conservation of energy for an invisicid fluid.

Example

Blood flow from the left ventricle into the aorta can be modeled as a reducing nozzle (see Figure 3.23). Model both the left ventricle and the aorta as a tube with diameter of 3.1 cm and

2.7 cm, respectively. The pressure in the left ventricle is 130 mmHg and the pressure in the aorta is 123 mmHg. Blood is ejected from the left ventricle at a speed of 120 cm/s. Calculate the difference in height between these two locations.

Solution

To solve this problem, we need to assume that the flow is steady, incompressible, and invisicid. Apply the Conservation of Mass to determine the blood velocity within the aorta.

$$0 = \int_{\text{area}} \rho \vec{v} \cdot d\vec{A} = -\rho v_1 A_1 + \rho v_2 A_2$$

$$v_2 = \frac{v_1 A_1}{A_2} = \frac{120\ \text{cm/s} * \pi\left(\frac{3.1\ \text{cm}}{2}\right)^2}{\pi\left(\frac{2.7\ \text{cm}}{2}\right)^2} = 158\ \text{cm/s}$$

Using Bernoulli to solve for the difference in height,

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} + gz_1 = \frac{v_2^2}{2} + \frac{p_2}{\rho} + gz_2$$

$$g(z_2 - z_1) = \frac{v_1^2}{2} + \frac{p_1}{\rho} - \frac{v_2^2}{2} - \frac{p_2}{\rho}$$

$$z_2 - z_1 = \frac{\frac{v_1^2}{2} + \frac{p_1}{\rho} - \frac{v_2^2}{2} - \frac{p_2}{\rho}}{g}$$

$$= \frac{\frac{(120\ \text{cm/s})^2}{2} + \frac{130\ \text{mmHg}}{1050\ \ \text{kg/m}^3} - \frac{(158\ \text{cm/s})^2}{2} - \frac{123\ \text{mmHg}}{1050\ \ \text{kg/m}^3}}{9.81\ \ \text{m/s}^2} = 3.68\ \text{cm}$$

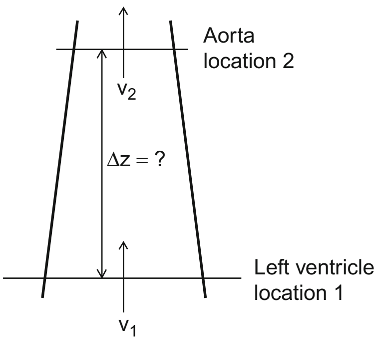

FIGURE 3.23 Schematic of the aorta downstream to the left ventricle. The aorta would experience a slight contraction within this area.

As you can imagine from the previous example, using the Bernoulli equation and the Conservation of Mass equations, we can estimate velocity, pressure, or height of a particular fluid. We want to emphasize that this is only an estimate in most real cases, because we need to make the assumption that the flow is invisicid to apply Bernoulli. Use caution when applying this powerful relationship. In most situations, if the blood vessel is large enough (e.g., aorta or vena cava), the Bernoulli equations can be applied, but as the vessel diameter reduces, the viscous forces play a more critical role in the flow. Therefore, the Bernoulli equations cannot be used in these situations and the Navier-Stokes equations, the Conservation of Momentum, and the Conservation of Mass should be applied.

END OF CHAPTER SUMMARY

3.1 The body forces acting on a differential fluid element are $\vec{F_b} = \vec{g}\,dm = \vec{g}\,\rho dxdydz$. The surface forces acting on a differential fluid element are $d\vec{F_s} = -\nabla\vec{p}\,dxdydz$. For static fluids, where all acceleration terms are zero, the pressure gradient is equal to the gravitational acceleration multiplied by the fluid density. In Cartesian components, this is

$$-\frac{\partial p}{\partial x} + \rho g_x = 0$$

$$-\frac{\partial p}{\partial y} + \rho g_y = 0$$

$$-\frac{\partial p}{\partial z} + \rho g_z = 0$$

Most pressures that are recorded in biofluids are gauge pressures, which can be defined as

$$p_{\text{gauge}} = p_{\text{absolute}} - p_{\text{atmospheric}}$$

3.2 Buoyancy is the net vertical force that acts on a floating or an immersed object. The buoyancy forces can be defined as

$$F_z = \int_V dF_z = \int_V \rho g dV = \rho g V$$

3.3 A generalized formulation for the time rate of change of a system property can be represented as

$$\frac{dW}{dt} = \frac{\partial}{\partial t}\int_V w\rho dV + \int_{\text{area}} w\rho\vec{v}\cdot d\vec{A}$$

Applying this formulation to the Conservation of Mass, we would get

$$\left.\frac{dm}{dt}\right|_{\text{system}} = \frac{\partial}{\partial t}\int_V \rho dV + \int_{\text{area}} \rho\vec{v}\cdot d\vec{A} = 0$$

Depending on the particular flow conditions, the conservation of mass formula can be simplified in various ways.

3.4 The Conservation of Momentum can be represented as

$$\frac{dP}{dt} = \vec{F} = \vec{F}_b + \vec{F}_s = \frac{\partial}{\partial t}\int_V \vec{v}\rho dV + \int_{\text{area}} \vec{v}\rho\vec{v}\cdot d\vec{A}$$

Again, this can be simplified depending on the particular flow conditions.

3.5 The Conservation of Momentum could also include acceleration components. In Cartesian component form this would be

$$\vec{F}_{bx} + \vec{F}_{sx} - \int_V \vec{a}_{rx} \rho dV = \frac{\partial}{\partial t}\int_V u_{xyz}\rho dV + \int_{\text{area}} u_{xyz}\rho \vec{v}_{xyz} \cdot d\vec{A}$$
$$\vec{F}_{by} + \vec{F}_{sy} - \int_V \vec{a}_{ry} \rho dV = \frac{\partial}{\partial t}\int_V v_{xyz}\rho dV + \int_{\text{area}} v_{xyz}\rho \vec{v}_{xyz} \cdot d\vec{A}$$
$$\vec{F}_{bz} + \vec{F}_{sz} - \int_V \vec{a}_{rz} \rho dV = \frac{\partial}{\partial t}\int_V w_{xyz}\rho dV + \int_{\text{area}} w_{xyz}\rho \vec{v}_{xyz} \cdot d\vec{A}$$

3.6 It is common for heat exchange to occur within biological flows. To account for the conservation of energy within a fluid, the formulation would become

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}\int_V e\rho dV + \int_{\text{area}} e\rho\vec{v} \cdot \left|d\vec{A}\right| = \frac{\partial}{\partial t}\int_V \left(u + \frac{v^2}{2} + gz\right)\rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz\right)\rho\vec{v} \cdot d\vec{A}$$

The rate of work in this equation would need to account for all of the various work terms that may be applied to the fluid. In a simplified form, this would be

$$\dot{Q} + \int_{\text{area}} \vec{\tau} \cdot \vec{v} dA - \dot{W}_{\text{shaft}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_V \left(u + \frac{v^2}{2} + gz\right)\rho dV + \int_{\text{area}} \left(u + \frac{v^2}{2} + gz + p\right)\rho\vec{v} \cdot d\vec{A}$$

The second law of thermodynamics can be applied to biofluids. It is represented as

$$\frac{\partial}{\partial t}\int_V s\rho dV + \int_{\text{area}} s\rho\vec{v} \cdot d\vec{A} \geq \frac{\dot{Q}}{T}$$

3.7 The Navier-Stokes equations are the solutions of Newton's Second Law of Motion applied to fluid flow. For incompressible flows with a constant viscosity, these equations simplify to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

in Cartesian coordinates and

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$
$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = \rho g_\theta - \frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

in cylindrical coordinates. From either of these sets of equations, the fully developed fluid velocity profile can be directly solved for as a function of location within the flow field.

3.8 The Bernoulli equation is a useful formula that relates the pressure variation in a fluid to the height and the speed of the fluid element. However, this formulation is only valid for steady, incompressible, and invisicid flows. The Bernoulli equation states that

$$\frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant}$$

HOMEWORK PROBLEMS

3.1 A two-fluid manometer is used to measure the pressure difference for flowing blood in a laboratory experiment (see Figure 3.24). Calculate the pressure difference between points A and B in the fluid.

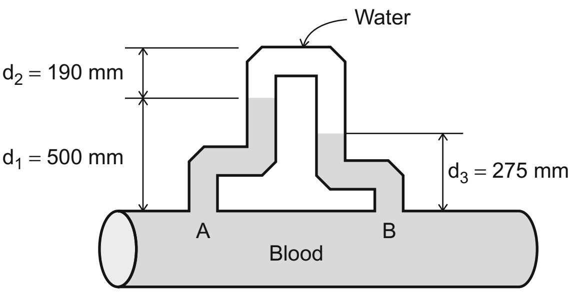

FIGURE 3.24 Figure for homework problem 3.1.

3.2 NASA is planning a mission to a newly found planet and will monitor the density of the new planet's atmosphere. Assume that NASA knows that atmosphere behaves as an ideal gas and that the planet's gravitational force is a function of altitude$\left(g(z) = 18.7\frac{\text{m}}{\text{s}^2}\left(1 - \frac{z}{10{,}000\ \text{m}}\right)\right)$, where z is in m). The temperature of the atmosphere is constant at 250 K, and the gas constant is 340 Nm/kgK. Assume that the pressure at the planet's surface is 2 atm. Calculate the pressure and density at an altitude of 1 km, 5 km, and 9 km.

3.3 Calculate the hydrostatic pressure in the cranium and in the feet at the end of systole and the end of diastole for a hypertensive patient (end systolic pressure is equal to 185 mmHg and end diastolic pressure is equal to 145 mmHg). Assume that the blood density does not change significantly with height and that the cranium is 25 cm above the aortic valve and the feet are 140 cm below the aortic valve. Compare this with a normal patient.

3.4 A balloon catheter has been placed within a femoral artery of a patient, to be passed to the coronary artery (use the same dimensions stated with Figure 3.6). Assume that the catheter consists of two components: 1) a chamber to hold the balloon, which is 2 mm in diameter and 1 cm in length (a perfect cylinder) and 2) a tube 0.5 mm in diameter and the total length needed to transport the balloon to the opening locations. Calculate the buoyancy force on this catheter.

3.5 Consider the steady, incompressible blood flow through the vascular network as shown. Determine the magnitude and the direction of the volume flow rate through the daughter branch 2 (denoted as D_3 in Figure 3.25).

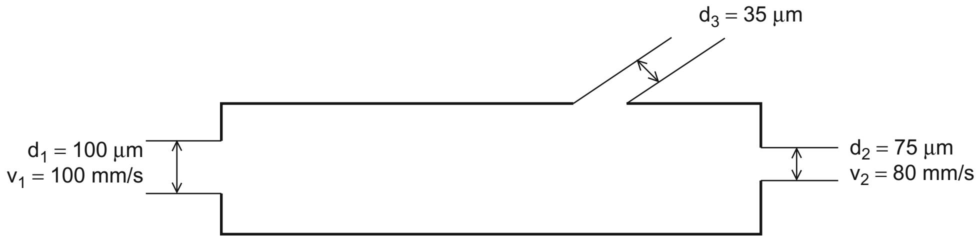

FIGURE 3.25 Figure for homework problem 3.5.

3.6 A biofluid flows with a density of 1080 kg/m^3 through the converging network as shown in Figure 3.26. Given that $d_1 = 15$ μm, $d_2 = 9$ μm, and $d_3 = 24$ μm, with $v_1 = 5$ mm/s $\vec{i}$ and $v_2 = 8$ mm/s $\vec{j}$, determine the velocity v_3.

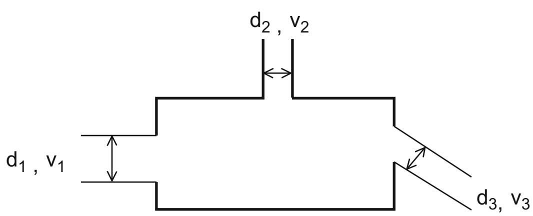

FIGURE 3.26 Figure for homework problem 3.6.

3.7 Using the same details for problem 3.6, calculate the change in time rate of change of volume if v_3 is equal to 10 mm/s.

3.8 Air enters the lungs through a circular channel with a diameter of 3 cm and a velocity of 150 cm/s and a density of 1.25 kg/m^3. Air leaves the lungs through the same opening at a velocity of 120 cm/s and a density equal to that of the lungs. At the initial conditions the air within the lungs has a density of 1.4 kg/m^3, with a total volume of 6 L. Find the initial rate of change of the density of air in the lung assuming that your time step includes one inhale and one exhale (takes 15 sec).

3.9 During peak systole, the heart delivers to the aorta a blood flow that has a velocity of 100 cm/sec at a pressure of 120 mmHg. The aortic root has a mean diameter of 25 mm. Determine the force acting on the aortic arch if the conditions at the outlet are a pressure of 110 mmHg and a diameter of 21 mm (see Figure 3.27). The density of blood is 1050 kg/m^3.

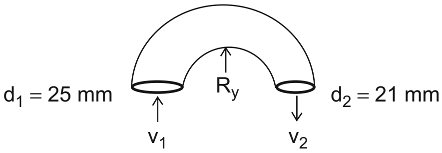

FIGURE 3.27 Figure for homework problem 3.9.

3.10 A reducing blood vessel has a 30° bend in it. Evaluate the components of force that must be provided by the adjacent tissue to keep the blood vessel in place. All necessary information is provided in Figure 3.28.

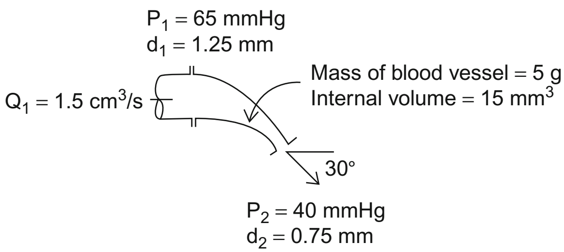

FIGURE 3.28 Figure for homework problem 3.10.

3.11 The following segment of the carotid artery (see Figure 3.29) has an inlet velocity of 50 cm/s (diameter of 15 mm). The outlet has a diameter of 11 mm. The pressure at the inlet is 110 mmHg and at the outlet is 95 mmHg. Determine the reaction forces to keep this vessel in place.

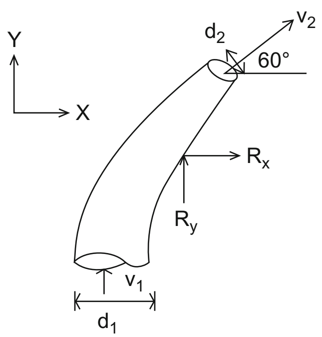

FIGURE 3.29 Figure for homework problem 3.11.

3.12 One of the first implantable mechanical heart valves was designed as a ball within a cage that acted as a check valve. Using the conservation of momentum (with acceleration), model the acceleration of the ball after it is hit by a jet of blood, being ejected from the heart (Figure 3.15). The ball has a turning angle of 45° and a mass of 20 g. Blood is ejected from the heart at a velocity of 110 cm/s, through an opening with a diameter of 25 mm. Determine the velocity of the ball at 0.5 sec. Neglect any resistance to motion (except mass).

3.13 During systole, blood is ejected from the left ventricle at a velocity of 125 cm/s. The diameter of the aortic valve is 24 mm, and there is no heat transfer or temperature change within the system. Assume that systole lasts for 0.25 sec, that the height difference is 5 cm, and that there is no change in area within this distance. Determine the amount of work performed by the heart during systole and the power that the heart generates.

3.14 Air at standard atmosphere conditions (1 atm and 25°C) enters the lungs at 50 cm/s and leaves at a pressure of 1.1 atm, 37°C, and a velocity of 60 cm/s (with a constant mass flow rate of 1.2 g/s). The body removes heat from the lungs at a rate of 15 J/g. Calculate the power required by the lungs.

3.15 The left common coronary artery has an axisymmetric constriction because of a plaque buildup (see Figure 3.30). Given the upstream conditions of a velocity of 20 cm/s (systole) and 12 cm/s (diastole), calculate the velocity at the stenosis throat and the pressure difference between the stenosis throat and the inlet.

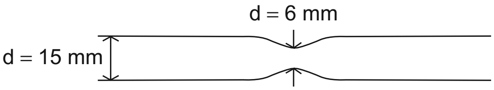

FIGURE 3.30 Figure for homework problem 3.15.

3.16 Blood flows through a 25% restricting (diameter reduces by 25%) blood vessel that experiences a 5 cm vertical drop (see Figure 3.31). The blood pressure at the inlet is 65 mmHg, and the blood velocity is 50 cm/s. Calculate the blood velocity the pressure at the outlet.

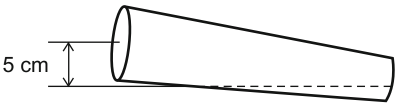

FIGURE 3.31 Figure for homework problem 3.16.

3.17 The cross-sectional area of a diverging vein may be expressed as $A = A_1 e^{ax}$, where A_1 is the cross-sectional area of that inlet. Develop a relationship for the velocity profile within the vein (in terms of x, v_1). Also, develop a relationship for the pressure (if the inlet pressure is p_1) in terms of x. Assume that there is no variation in height.

3.18 Blood flows through a vertical tube with a kinematic viscosity of 3×10^{-4} m/s^2 by gravity only (see Figure 3.32). Solve the appropriate Navier-Stokes equations to find the velocity distribution $v_Z(r)$ and compute the average velocity.

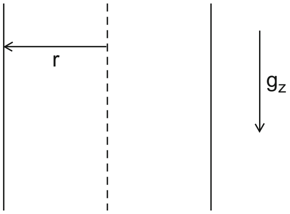

FIGURE 3.32 Figure for homework problem 3.18.

3.19 Solve problem 3.18 assuming that the blood is flowing within a vertical parallel plate (i.e., calculate with the Cartesian Navier-Stokes equations), where the coordinate system is aligned with the wall and channel width is h (see Figure 3.33).