Intro to Markov Chains

Stochastic (Random) Processes

- Some events occur under probability, for example in weather forecast (rainfall, storm etc), stock markets
- stochastic processes are processes that are random.

The Markov chain is a stochastic process. It is discrete-time i.e. not continuous in time. It is also discrete-state i.e. not continuous in occurrence.

The Markov Reliability Model

Systems or components - particularly large & complex ones - can possibly be in any of ~~the~~ a number of functioning states. These states include fully functioning, degraded and totally failed states. A system in a degraded state has not totally failed nor fully functioning, rather it functions only partially. An example is a server that provides internet service to a number of computer systems on an intranet work. If the server goes down, the computers will not have access to the internet but still would perform some individual tasks. Thus, the network here is considered degraded since only the task(s) that the computer terminals would require the server for could not be achieved.

It is possible for systems or components to transit from one state to another. An example of a transition from a degraded state to a fully functional state is when maintenance/repair work is carried out on the server mentioned earlier.

A system, for example a computer that undergoes necessary repairs ~~for~~ ~~pet~~ can transit from a failed state to a fully functional state. failure of the power supply unit of an electronic equipment can cause the fully-functioning equipment to transit to a totally failed state. If this transitions are observed over a reasonably long period of time, it is possible to have an idea of the probability of transition from one state to the other. These probability values, with appropriate analysis can be used to determine the probability of the system being in one state or the other at a future time. This is the Markov reliability model.

The model can also be adopted for analysing the reliability of repairable systems. It can be used to assess the effect of a specified repair capability on system availability & reliability.

Assumptions of Markov Analysis

Markov suggests that state-space analysis is the best known degraded reliability analysis. This analysis can be applied under the following conditions:

1. Stationary: The probability of changing from one state to another must be kept constant. This means the process must be ~~stationary~~ homogeneous. This implies that Markov analysis can only be valid for a system/process operating under constant failure rate.

2. Markov Chain Assumption: This assumes that the transition process is memoryless. In other words, the

future states of a system are independent of all the past states, except the immediate preceding state.

Markov analysis can be carried out on both discrete- and continuous-time processes, depending on the nature of the problem at hand.

The Markov Modelling Process

This is illustrated in the flow diagram below:

Discrete-Time Markov Process (Markov Chains)

A process is random if it evolves over time in a manner that cannot be completely predicted i.e. the process occurs in a completely probabilistic manner.
A discrete-time stochastic process is any segment of random variables $X_1, X_2, X_3, \ldots X_n$ indexed by time.

The state is the process at time, n, that is X_n. A transition occurs when there is a change of state. Thus, a discrete-time Markov process is a discrete-time stochastic process that obeys the ~~two~~ stationary and Markov chain assumptions.

The example below illustrates the application of the discrete-state discrete-time Markov process i.e. Markov chain to solve a problem of degraded reliability.

Example 1:

Consider a system that can be in any of the three states 1, 2, and 3 at present, and can be in any of the three states in a future time (the most immediate time). ~~Table~~ The table shows the probability of transiting from a present to a future state.

Table 1: Transition Probability Data

Present state	Possible future states		
	1	2	3
1	0.5	0.5	0.0
2	0.0	0.5	0.5
3	0.75	0.25	0.0

Task:

(i) Obtain the one-step transition matrix

(ii) Sketch the one-step transition diagram.

Solution:

i. This is an example of a discrete-time discrete-process. The one-step transition matrix is a matrix whose components are the probabilities of transition from ~~one~~ a present state to a future state in a unit time (or step). The present and future states ~~are~~ occupy the rows and collumns respectively. Thus, the one-step transition matrix is:

$$P = \begin{bmatrix} 0.50 & 0.50 & 0.00 \\ 0.00 & 0.50 & 0.50 \\ 0.75 & 0.25 & 0.00 \end{bmatrix} = [P_{ij}]$$

Inferences from the matrix:

(a) It is a square matrix

(b) All entries are between 0 and 1 since they are probability values.

(c) The rows add up to unity since there can be transition from one state to any of the three states (including 'remaining in the previous state').

ii. The one-step transition diagram is a pictorial map of the process in which states are represented by points (or nodes) and transitions by arrows.

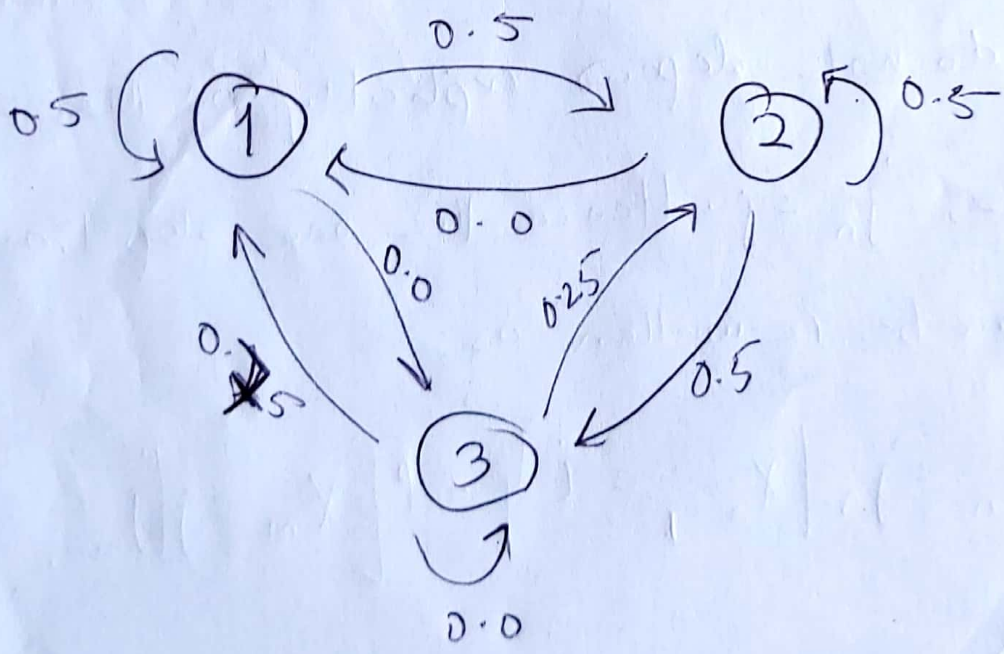

Figure: Transition Diagram for the Example

Markov Theory

Given $X_n - 1$, a discrete-state discrete-time Markov process can be described by the conditional distribution of random variable X_n.

$$P\{X_n = j_n \mid X_{n-1}, X_{n-2} = j_{n-2}, \ldots, X_1 = j_1, X_0 = j_0\} =$$

$$P\{X_n = j_n \mid j_{n-1}\} \quad \text{—— (i)}$$

The RHS of the equation is known as the one-step transition probability from X_{n-1} to X_n at step n.

For a stationary Markov chain, the transition probabilities do not depend explicitly on the step number. Thus, $j_n = j$. Hence the one-step transition probability can be re-written as:

$$P\{X_n = j_n \mid X_{n-1} = i\} = P\{X_m = j \mid j_{m-1} = i\} \quad \text{—— (ii)}$$

for all m and n parameters.

Since i and j represent the immediate past and present states respectively, the expression can be more precisely written as:

$$P_{ij} = P\left[X_1 = j \mid X_0 = i\right] \qquad \text{——— (iii)}$$

Equation iii can be explained as the one-step transition probability from state i to state j in one step. For a given process, the P_{ij}s form a square matrix P given by:

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3n} \\ \vdots & \vdots & \vdots & & \\ P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} \end{bmatrix} \qquad \text{——— (iv)}$$

P is referred to as the one-step transition matrix, with the element P_{ij} representing the probability of transition from state i to state j in one step.

Steady - state Probabilities

The one-step transition matrix for the three-state problem in example 1 can be generally written as:

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \quad \text{——— (v)}$$

where P_{ij} is the conditional probability of transition from state i to state j in one step as earlier explained. i.e.

$$P_{ij} = P\left[X_1 = j / X_0 = i\right], \; i,j = 1,2,3 \quad \text{——— (vi)}$$

Since the system is assumed stationary, the one-step transition matrix P is sufficient to describe the entire process. If transition matrix P is regular, there exists powers of P. A transition matrix is termed regular if some power of the matrix, P^x (x is an integer) contains all positive entries. Hence, a Markov chain is a regular Markov chain if its Transition matrix is regular. This gives the two-step transition matrix as:

$$P^{(2)} = P \cdot P = P^2 = \begin{pmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{pmatrix} \quad \text{——(vii)}$$

where $P_{ij}^{(2)}$ is the probability of transiting from state i to state j in two steps.

Similarly,

$$P^{(3)} = P^{(2)} \cdot P = P^2 \cdot P = P^3 = \begin{pmatrix} P_{11}^{(3)} & P_{12}^{(3)} & P_{13}^{(3)} \\ P_{21}^{(3)} & P_{22}^{(3)} & P_{23}^{(3)} \\ P_{31}^{(3)} & P_{32}^{(3)} & P_{33}^{(3)} \end{pmatrix} \quad \text{——(viii)}$$

And the n-step transition matrix is :

$$P^{(n)} = P^{(n-1)} \cdot P = P^n = \begin{pmatrix} P_{11}^{(n)} & P_{12}^{(n)} & P_{13}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & P_{23}^{(n)} \\ P_{31}^{(n)} & P_{32}^{(n)} & P_{33}^{(n)} \end{pmatrix} \quad \text{--- (ix)}$$

Typically, such processes stabilise as n approaches infinity i.e. after a long period of time. In such cases, the 'present state' tends to lose significance and can be termed steady-state probabilities, or equilibrium probability, or limiting values. ~~while~~

Under such condition,

$$\pi_j = \lim_{n \to \infty} P_j^{(n)} = \lim_{n \to \infty} P\{X = j\} \quad \text{--- (x)}$$

this implies that $P^{(n)}$ has identical rows.

$$\bar{\pi} = \lim_{n=\infty} P^{(n)} = \lim_{n\to\infty} P^{(n-1)} \cdot P = \begin{bmatrix} \bar{\pi}_1 & \bar{\pi}_2 & \bar{\pi}_3 \\ \bar{\pi}_1 & \bar{\pi}_2 & \bar{\pi}_3 \\ \vdots & & \end{bmatrix} \qquad \text{———— (xi)}$$

i.e.

$$\bar{\pi} = \bar{\pi} \cdot P \qquad \text{———— (xii)}$$

where

$$\bar{\pi} = (\bar{\pi}_1 \;\; \bar{\pi}_2 \;\; \bar{\pi}_3) \qquad \text{———— (xiii)}$$

Therefore, $\bar{\pi}_j$ is the steady-state probability of finding the system in state j. Or, $\bar{\pi}_j$ is the probability that the system will be in state j after a long time.

Every row of equation (xi) must add up to unity. i.e. the normalised equation is:

$$\sum_{i=1}^{n} \bar{\pi}_i = 1 \qquad \text{———— (xiv)}$$

This gives a dependent set of equations. An additional equation that is therefore required to solve the simultaneous equations is the normalizing equation (x:v). i.e. $\bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3 = 1$

Example 2:

Recall example 1,

(i) Determine the steady-state probabilities of the system being in each of the three states.

(ii) Calculate the mean time spent in each state.