se

all from double-angle formular

$\cos(A+A) = \cos A\cos A - \sin A\sin A$

$\cos 2A = \cos^2 A - \sin^2 A$ —— (1)

And

$\cos^2 A + \sin^2 A = 1$

$\cos^2 A = 1 - \sin^2 A$ —— (2)

$\sin^2 A = 1 - \cos^2 A$ —— (3)

Put (2) in (1)

$\cos 2A = 1 - \sin^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

$2\sin^2 A = 1 - \cos 2A$

$$\boxed{\sin^2 A = \frac{1-\cos 2A}{2}}$$

Also, put (3) in (1)

$\cos 2A = \cos^2 A - 1 + \cos^2 A$

$\cos 2A = 2\cos^2 A - 1$

$$\boxed{\cos^2 A = \frac{1+\cos 2A}{2}}$$

Example 1

$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\int (1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

$= \frac{x}{2} - \frac{\sin 2x}{4} + c$.

Example 2

$\int \cos^2 x\,dx = \int\left(\frac{1+\cos 2x}{2}\right)dx$

$= \frac{1}{2}\int (1+\cos 2x)\,dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] = \frac{x}{2} + \frac{\sin 2x}{4} + c$

(3)

$$\left.\begin{aligned} &\int \sin^3 x\,dx \\ &= \int \sin x \cdot \sin^2 x\,dx \\ &= \int \sin x\,(1-\cos^2 x) \end{aligned}\right\}\text{Later.}$$

{when one power is odd and other even}

Note

The integral $\int \sin^m x \cos^n x\,dx$ can be evaluated easily, if m or n is an odd integers. If m is odd, the substitution $u = \cos x$ is used; if n is odd, the substitution $u = \sin x$ is used

Example 1

$\int \sin^3 x \cos^2 x\,dx$

Since m is odd,

$u = \cos x$

$\frac{du}{dx} = -\sin x \Rightarrow dx = \frac{-du}{\sin x}$

And

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$= \int \sin x \cdot \sin^2 x \cdot u^2 \cdot \frac{-du}{\sin x}$

$= -\int \sin^2 x \cdot u^2\,du$

$= -\int (1-\cos^2 x)\cdot u^2\,du$

$= \int (u^2-1)u^2\,du$

$= \int (u^4 - u^2)\,du$

$= \left[\frac{u^5}{5} - \frac{u^3}{3}\right] + c$

$= \frac{(\cos x)^5}{5} - \frac{(\cos x)^3}{3} + c$

(2) $\int \frac{\cos^3 x}{\sin^6 x} dx$

Let $u = \sin x$

$\frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$

$\cos^2 x = 1 - \sin^2 x$

we have

$= \int \frac{\cos x \cdot \cos^2 x}{u^6} \cdot \frac{du}{\cos x}$

$= \int \frac{1-\sin^2 x}{u^6} du = \int \frac{1-u^2}{u^6} du$

$\int \left(\frac{1}{u^6} - \frac{u^2}{u^6}\right) du$

$= \int (u^{-6} - u^{-4}) du$

$= \frac{u^{-5}}{-5} - \frac{u^{-3}}{-3} + c$

$= \frac{u^{-3}}{3} - \frac{u^{-5}}{5}$

$= \frac{(\sin x)^{-3}}{3} - \frac{(\sin x)^{-5}}{5}$

(4)

$\int \cos^4 x \, dx$

since

$\cos^2 x = \frac{1}{2}[1 + \cos 2x]$

$\cos^4 x = \left(\frac{1 + \cos 2x}{2}\right)^2$

$\frac{1}{4} \int (1 + \cos 2x)^2$

$\frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x$

And since

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$\cos^2 2x = \frac{1 + \cos 2(2x)}{2} = \frac{1 + \cos 4x}{2}$

we have

$\frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2}\right) dx$

$\frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx$

$\frac{1}{4}\left[\frac{3}{2}x + \frac{2\sin 2x}{2} + \frac{\sin 4x}{8}\right] + c$

$= \frac{3x}{8} + \frac{2\sin 2x}{\cancel{8}\,4} + \frac{\sin 4x}{32} + c$

$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$

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Example 1

$$\int \sin^2 x \cos^2 x \, dx$$

Since $\sin^2 x = 1 - \cos^2 x$

$$\int (1-\cos^2 x)\cos^2 x \, dx$$

$$\int (\cos^2 x - \cos^4 x) \, dx$$

$$\int \cos^2 x \, dx - \int \cos^4 x \, dx$$

$$\frac{x}{2} + \frac{\sin 2x}{4} - \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right] + c$$

when both powers are even we have

Example 1

$$\Rightarrow \frac{x}{8} - \frac{\sin 4x}{32} + c$$

NB

We have the same value if we use $1 - \sin^2 x = \cos^2 x$.

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Example 2

$$\int \sin^2 t \cos^4 t\,dt$$

$$\int \sin^2 t\,(\cos^2 t)^2\,dt$$

$$\int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$\int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+2\cos 2t+\cos^2 2t}{4}\right) dt$$

$$=\frac{1}{8}\int \left[1+2\cos 2t+\cos^2 2t-\cos 2t-2\cos^2 2t-\cos^3 2t\right]$$

$$=\frac{1}{8}\int \left[1+\cos 2t-\cos^2 2t-\cos^3 2t\right]dt$$

$$=\frac{1}{8}\int 1+\cos 2t-\left[\frac{1+\cos 4t}{2}\right]-\cos 2t\,(1-\sin^2 2t)\,dt$$

$$=\frac{1}{8}\int \frac{1}{2}-\frac{\cos 4t}{2}+\cos 2t\sin^2 2t\,dt$$

$$\frac{1}{8}\left[\frac{t}{2}-\frac{\sin 4t}{8}+\frac{\sin^3 2t}{6}\right]+c$$

since

$$\int \cos 2t\sin^2 2t\,dt \text{ gives}$$

Let $u=\sin 2t$

$$\frac{du}{dt}=2\cos 2t$$

$$du=2\cos 2t\,dt$$

$$\frac{du}{2}=\cos 2t\,dt$$

$$\Rightarrow \int u^2\frac{du}{2}=\frac{1}{2}\int u^2\,du$$

$$=\frac{u^3}{6}+c=\frac{\sin^3 2t}{6}+c$$

{when both powers are odd}

(1)

$$\int \sin x\cos^3 x\,dx$$

$$u=\cos x$$

$$\frac{du}{dx}=-\sin x$$

$$du=-\sin x\,dx$$

$$-du=\sin x\,dx$$

$$\int u^3\cdot -du=-\int u^3\,du$$

$$=-\left[\frac{u^4}{4}\right]+c$$

$$\Rightarrow \frac{-\cos^4 x}{4}+c$$

(2)

$$\int \sin^3 x\cos^5 x\,dx$$

$$u=\cos x$$

$$\frac{du}{dx}=-\sin x$$

$$du=-\sin x\,dx$$

$$dx=\frac{-du}{\sin x}$$

$$\int \sin^3 x\cdot u^5\cdot\frac{-du}{\sin x}$$

$$-\int \sin^2 x\cdot u^5\,du$$

$$-\int (1-\cos^2 x)\cdot u^5\,du$$

$$-\int (1-u^2)u^5\,du$$

$$\int (-u^5+u^7)\,du$$

$$=\frac{u^8}{8}-\frac{u^6}{6}+c$$

$$=\frac{\cos^8 x}{8}-\frac{\cos^6 x}{6}+c$$

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(4)

$$\int \sin^3 x \, dx$$

Recall $\sin^2 x = 1 - \cos^2 x$

$$= \int \sin x (\sin^2 x) dx$$

$$= \int \sin x (1 - \cos^2 x) dx$$

$$\int (\sin x - \sin x \cos^2 x) dx$$

$$= \left[-\cos x + \frac{\cos^3 x}{3} \right] + C$$

(5)

$$\int \sin^5 \theta \, d\theta$$

Also, $\sin^2\theta = 1 - \cos^2\theta$

$$= \int \sin\theta (\sin^2\theta)^2 d\theta$$

$$= \int \sin\theta (1 - \cos^2\theta)^2 d\theta$$

$$= \int \sin\theta (1 - 2\cos^2\theta + \cos^4\theta) d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta) d\theta$$

$$= -\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + C$$

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DEFINITE INTEGRALS

Integrals containing an arbitrary constant C in their results are called indefinite integrals, since their precise value cannot be determined without further information.

Definite integrals are those in which limits are applied. If an expression is written as $[x]_a^b$, 'b' is called the upper limit and 'a' the lower limit.

The operation of applying the limits is defined as

$$\left[x\right]_a^b = (b) - (a)$$

For example

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}$$

$$= 8\frac{2}{3}.$$

$$\int_1^2 4e^{2x} dx =$$

$$= \left[\frac{4e^{2x}}{2}\right]_1^2 = \left[2e^{2x}\right]_1^2$$

$$= 2e^{2(2)} - 2e^{2(1)} = 2e^4 - 2e^2$$

$$= 94.42$$

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Definite Integrals (Examples)

① $\int_{-2}^{3}(4-x^2)\,dx$

$\Rightarrow \left[4x-\frac{x^3}{3}\right]_{-2}^{3}$

$=\left[4(3)-\frac{3^3}{3}\right]-\left[4(-2)-\frac{(-2)^3}{3}\right]$

$=\left[12-9\right]-\left[\frac{-8}{1}+\frac{8}{3}\right]$

$=\left[3\right]-\left[\frac{-24+8}{3}\right]=\frac{3}{1}+\frac{16}{3}$

$=\frac{9+16}{3}=\frac{25}{3}=8\frac{1}{3}$

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Double Integral

1.

$$\int_0^2 \left\{ \int_1^3 \left[xy + x^2y^3 \right] dx \right\} dy$$

$$\int_1^3 (xy + x^2y^3)\,dx$$

$$= \left[\frac{x^2}{2}y + \frac{x^3}{3}y^3 \right]_1^3$$

$$= \left[\frac{9y}{2} + 9y^3 \right] - \left[\frac{y}{2} + \frac{y^3}{3} \right]$$

$$= 4y + 9y^3 - \frac{y^3}{3} \Rightarrow 4y + \frac{26y^3}{3}$$

$$= \int_0^2 \left(4y + \frac{26y^3}{3} \right) dy$$

$$= \left[\frac{4y^2}{2} + \frac{\overset{13}{26}y^4}{\underset{6}{12}} \right]_0^2$$

$$= \left[2y^2 + \frac{13y^4}{6} \right]_0^2$$

$$= \frac{2(2)^2}{1} + \frac{13(2)^4}{6} = \frac{8}{1} + \frac{208}{6}$$

$$= \frac{256}{6} = 42.67$$

2

2.

$$\int_1^2 \left[\int_0^3 x^2y\,dx \right] dy$$

$$\int_0^3 x^2y\,dx$$

$$\left[\frac{x^3y}{3} \right]_0^3$$

$$= \int_1^2 9y\,dy$$

$$= \left[\frac{9y^2}{2} \right]_1^2$$

$$= \frac{9(2)^2}{2} - \frac{9(1)^2}{2}$$

$$= \frac{36}{2} - \frac{9}{2} = \frac{27}{2}$$

$$= 13.5$$

3.

$$\int_1^2 \left[\int_0^3 (1 + 8xy)\,dx \right] dy$$

$$\int_0^3 (1 + 8xy)\,dx$$

$$\left[x + \frac{8x^2y}{2} \right]_0^3$$

$$= \int_1^2 (3 + 36y)\,dy$$

$$= \left[3y + \frac{36y^2}{2} \right]_1^2$$

$$\left[3y + 18y^2 \right]_1^2$$

$$[6 + 72] - [3 + 18]$$

$$78 - 21 = 57$$

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