SECTION 3
GENERAL STRUCTURAL THEORY

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Safety and serviceability constitute the two primary requirements in structural design. For a structure to be safe, it must have adequate strength and ductility when resisting occasional extreme loads. To ensure that a structure will perform satisfactorily at working loads, functional or serviceability requirements also must be met. An accurate prediction of the behavior of a structure subjected to these loads is indispensable in designing new structures and evaluating existing ones.

The behavior of a structure is defined by the displacements and forces produced within the structure as a result of external influences. In general, structural theory consists of the essential concepts and methods for determining these effects. The process of determining them is known as **structural analysis.** If the assumptions inherent in the applied structural theory are in close agreement with actual conditions, such an analysis can often produce results that are in reasonable agreement with performance in service.

3.1 FUNDAMENTALS OF STRUCTURAL THEORY

Structural theory is based primarily on the following set of laws and properties. These principles often provide sufficient relations for analysis of structures.

Laws of mechanics. These consist of the rules for static equilibrium and dynamic behavior.

Properties of materials. The material used in a structure has a significant influence on its behavior. Strength and stiffness are two important material properties. These properties are obtained from experimental tests and may be used in the analysis either directly or in an idealized form.

Laws of deformation. These require that structure geometry and any incurred deformation be compatible; i.e., the deformations of structural components are in agreement such that all components fit together to define the deformed state of the entire structure.

STRUCTURAL MECHANICS—STATICS

An understanding of basic mechanics is essential for comprehending structural theory. Mechanics is a part of physics that deals with the state of rest and the motion of bodies under the action of forces. For convenience, mechanics is divided into two parts: statics and dynamics.

Statics is that branch of mechanics that deals with bodies at rest or in equilibrium under the action of forces. In elementary mechanics, bodies may be idealized as rigid when the actual changes in dimensions caused by forces are small in comparison with the dimensions of the body. In evaluating the deformation of a body under the action of loads, however, the body is considered deformable.

3.2 PRINCIPLES OF FORCES

The concept of force is an important part of mechanics. Created by the action of one body on another, force is a vector, consisting of magnitude and direction. In addition to these values, point of action or line of action is needed to determine the effect of a force on a structural system.

Forces may be concentrated or distributed. A **concentrated force** is a force applied at a point. A **distributed force** is spread over an area. It should be noted that a concentrated force is an idealization. Every force is in fact applied over some finite area. When the dimensions of the area are small compared with the dimensions of the member acted on, however, the force may be considered concentrated. For example, in computation of forces in the members of a bridge, truck wheel loads are usually idealized as concentrated loads. These same wheel loads, however, may be treated as distributed loads in design of a bridge deck.

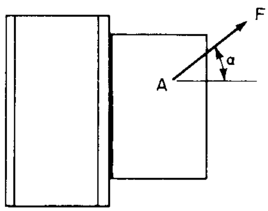


FIGURE 3.1 Vector **F** represents force acting on a bracket.

A set of forces is **concurrent** if the forces all act at the same point. Forces are **collinear** if they have the same line of action and are **coplanar** if they act in one plane.

Figure 3.1 shows a bracket that is subjected to a force **F** having magnitude F and direction defined by angle α. The force acts through point A. Changing any one of these designations changes the effect of the force on the bracket.

Because of the additive properties of forces, force **F** may be resolved into two concurrent force components $\mathbf{F}_x$ and $\mathbf{F}_y$ in the perpendicular directions x and y, as shown in Figure 3.2*a*. Adding these forces $\mathbf{F}_x$ and $\mathbf{F}_y$ will result in the original force **F** (Fig. 3.2*b*). In this case, the magnitudes and angle between these forces are defined as

$$F_x = F \cos \alpha \tag{3.1a}$$

$$F_y = F \sin \alpha \tag{3.1b}$$

$$F = \sqrt{F_x^2 + F_y^2} \tag{3.1c}$$

$$\alpha = \tan^{-1} \frac{F_y}{F_x} \tag{3.1d}$$

Similarly, a force **F** can be resolved into three force components $\mathbf{F}_x$, $\mathbf{F}_y$, and $\mathbf{F}_z$ aligned along three mutually perpendicular axes x, y, and z, respectively (Fig. 3.3). The magnitudes of these forces can be computed from

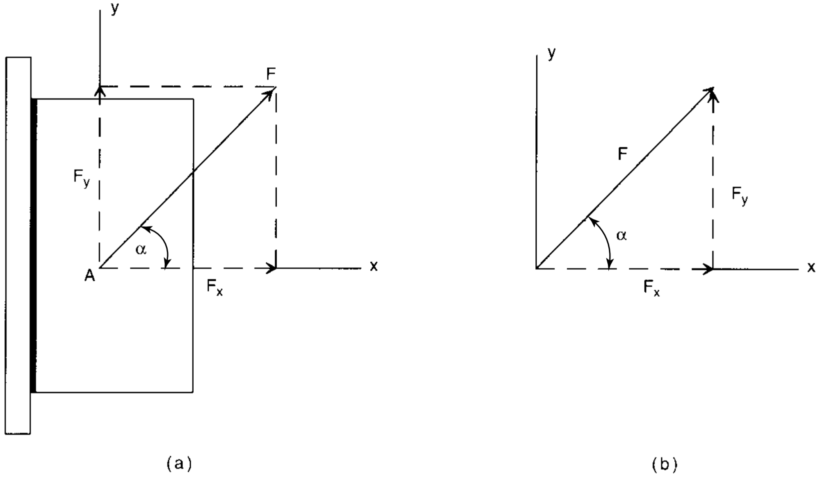


FIGURE 3.2 (*a*) Force **F** resolved into components, $\mathbf{F}_x$ along the *x* axis and $\mathbf{F}_y$ along the *y* axis. (*b*) Addition of forces $\mathbf{F}_x$ and $\mathbf{F}_y$ yields the original force **F**.

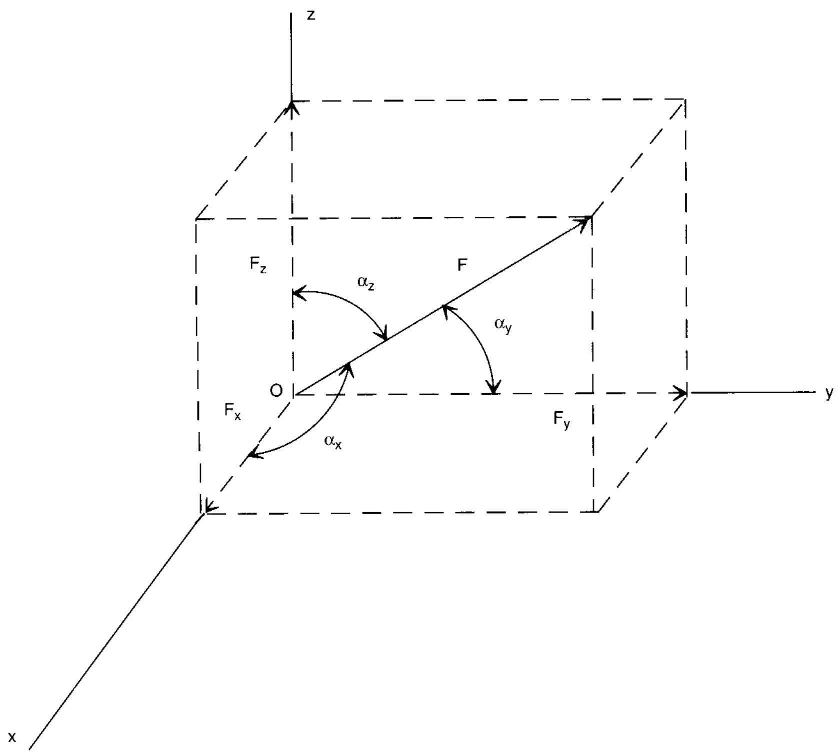


FIGURE 3.3 Resolution of a force in three dimensions.

$$F_x = F \cos \alpha_x \tag{3.2a}$$

$$F_y = F \cos \alpha_y \tag{3.2b}$$

$$F_z = F \cos \alpha_z \tag{3.2c}$$

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{3.2d}$$

where α_x, α_y, and α_z are the angles between **F** and the axes and $\cos \alpha_x$, $\cos \alpha_y$, and $\cos \alpha_z$ are the **direction cosines** of **F.**

The resultant **R** of several concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ (Fig. 3.4*a*) may be determined by first using Eqs. (3.2) to resolve each of the forces into components parallel to the assumed *x*, *y*, and *z* axes (Fig. 3.4*b*). The magnitude of each of the perpendicular force components can then be summed to define the magnitude of the resultant's force components $\mathbf{R}_x$, $\mathbf{R}_y$, and $\mathbf{R}_z$ as follows:

$$R_x = \Sigma F_x = F_{1x} + F_{2x} + F_{3x} \tag{3.3a}$$

$$R_y = \Sigma F_y = F_{1y} + F_{2y} + F_{3y} \tag{3.3b}$$

$$R_z = \Sigma F_z = F_{1z} + F_{2z} + F_{3z} \tag{3.3c}$$

The magnitude of the resultant force **R** can then be determined from

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \tag{3.4}$$

The direction **R** is determined by its direction cosines (Fig. 3.4*c*):

$$\cos \alpha_x = \frac{\Sigma F_x}{R} \qquad \cos \alpha_y = \frac{\Sigma F_y}{R} \qquad \cos \alpha_z = \frac{\Sigma F_z}{R} \tag{3.5}$$

where α_x, α_y, and α_z are the angles between *R* and the *x*, *y*, and *z* axes, respectively.

If the forces acting on the body are noncurrent, they can be made concurrent by changing the point of application of the acting forces. This requires incorporating moments so that the external effect of the forces will remain the same (see Art. 3.3).

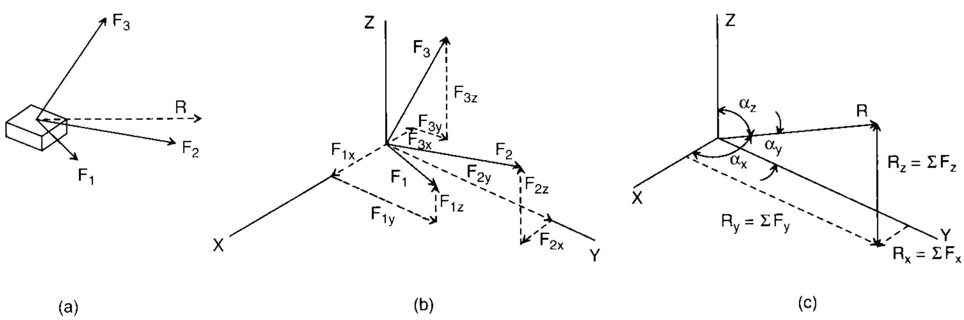


FIGURE 3.4 Addition of concurrent forces in three dimensions. (*a*) Forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ act through the same point. (*b*) The forces are resolved into components along *x*, *y*, and *z* axes. (*c*) Addition of the components yields the components of the resultant force, which, in turn, are added to obtain the resultant.

3.3 MOMENTS OF FORCES

A force acting on a body may have a tendency to rotate it. The measure of this tendency is the **moment** of the force about the axis of rotation. The moment of a force about a specific point equals the product of the magnitude of the force and the normal distance between the point and the line of action of the force. Moment is a vector.

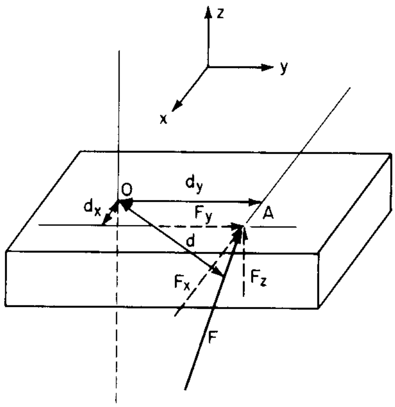


FIGURE 3.5 Moment of force **F** about an axis through point O equals the sum of the moments of the components of the force about the axis.

Suppose a force **F** acts at a point A on a rigid body (Fig. 3.5). For an axis through an arbitrary point O and parallel to the z axis, the magnitude of the moment **M** of **F** about this axis is the product of the magnitude F and the normal distance, or moment arm, d. The distance d between point O and the line of action of **F** can often be difficult to calculate. Computations may be simplified, however, with the use of **Varignon's theorem,** which states that the moment of the resultant of any force system about any axis equals the algebraic sum of the moments of the components of the force system about the same axis. For the case shown the magnitude of the moment **M** may then be calculated as

$$M = F_x d_y + F_y d_x \tag{3.6}$$

where F_x = component of **F** parallel to the x axis
F_y = component of **F** parallel to the y axis
d_y = distance of F_x from axis through O
d_x = distance of F_y from axis through O

Because the component F_z is parallel to the axis through O, it has no tendency to rotate the body about this axis and hence does not produce any additional moment.

In general, any force system can be replaced by a single force and a moment. In some cases, the resultant may only be a moment, while for the special case of all forces being concurrent, the resultant will only be a force.

For example, the force system shown in Figure 3.6*a* can be resolved into the equivalent force and moment system shown in Fig. 3.6*b*. The force **F** would have components F_x and F_y as follows:

$$F_x = F_{1x} + F_{2x} \tag{3.7a}$$

$$F_y = F_{1y} - F_{2y} \tag{3.7b}$$

The magnitude of the resultant force **F** can then be determined from

$$F = \sqrt{F_x^2 + F_y^2} \tag{3.8}$$

With Varignon's theorem, the magnitude of moment **M** may then be calculated from

$$M = -F_{1x} d_{1y} - F_{2x} d_{2y} + F_{1y} d_{2x} - F_{2y} d_{2x} \tag{3.9}$$

with d_1 and d_2 defined as the moment arms in Fig. 3.6*c*. Note that the direction of the

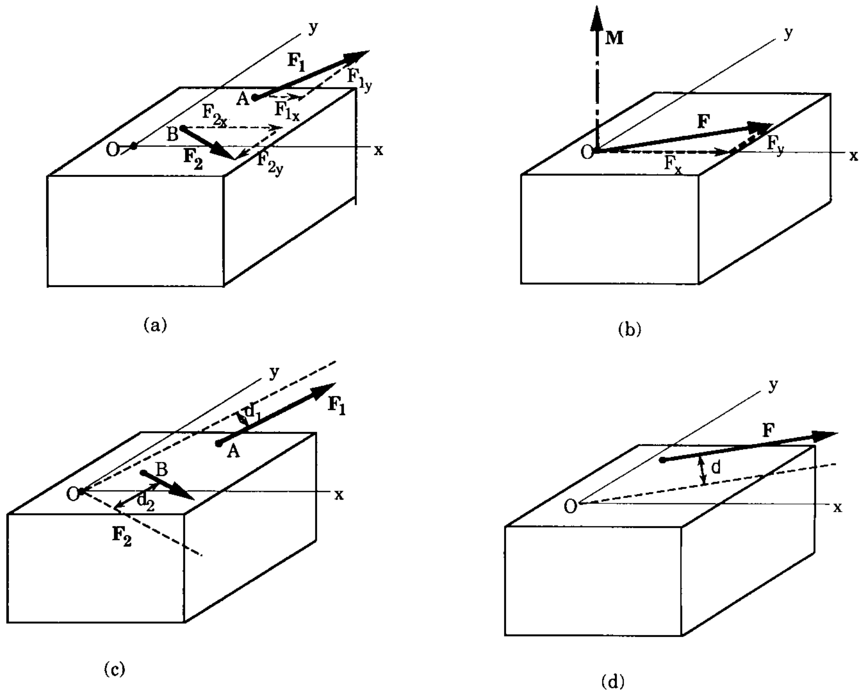


FIGURE 3.6 Resolution of concurrent forces. (*a*) Noncurrent forces $\mathbf{F}_1$ and $\mathbf{F}_2$ resolved into force components parallel to x and y axes. (*b*) The forces are resolved into a moment **M** and a force **F**. (*c*) **M** is determined by adding moments of the force components. (*d*) The forces are resolved into a couple comprising **F** and a moment arm d.

moment would be determined by the sign of Eq. (3.9); with a right-hand convention, positive would be a counterclockwise and negative a clockwise rotation.

This force and moment could further be used to compute the line of action of the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (Fig. 3.6*d*). The moment arm d could be calculated as

$$d = \frac{M}{F} \tag{3.10}$$

It should be noted that the four force systems shown in Fig. 3.6 are equivalent.

3.4 EQUATIONS OF EQUILIBRIUM

When a body is in **static equilibrium,** no translation or rotation occurs in any direction (neglecting cases of constant velocity). Since there is no translation, the sum of the forces acting on the body must be zero. Since there is no rotation, the sum of the moments about any point must be zero.

In a two-dimensional space, these conditions can be written:

$$\Sigma F_x = 0 \tag{3.11a}$$

$$\Sigma F_y = 0 \tag{3.11b}$$

$$\Sigma M = 0 \tag{3.11c}$$

where ΣF_x and ΣF_y are the sum of the components of the forces in the direction of the perpendicular axes x and y, respectively, and ΣM is the sum of the moments of all forces about any point in the plane of the forces.

Figure 3.7*a* shows a truss that is in equilibrium under a 20-kip (20,000-lb) load. By Eq. (3.11), the sum of the reactions, or forces R_L and R_R, needed to support the truss, is 20 kips. (The process of determining these reactions is presented in Art. 3.29.) The sum of the moments of all external forces about any point is zero. For instance, the moment of the forces about the right support reaction R_R is

$$\Sigma M = (30 \times 20) - (40 \times 15) = 600 - 600 = 0$$

(Since only vertical forces are involved, the equilibrium equation for horizontal forces does not apply.)

A **free-body diagram** of a portion of the truss to the left of section *AA* is shown in Fig. 3.7*b*). The internal forces in the truss members cut by the section must balance the external force and reaction on that part of the truss; i.e., all forces acting on the free body must satisfy the three equations of equilibrium [Eq. (3.11)].

For three-dimensional structures, the equations of equilibrium may be written

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{3.12a}$$

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \tag{3.12b}$$

The three force equations [Eqs. (3.12*a*)] state that for a body in equilibrium there is no resultant force producing a translation in any of the three principal directions. The three moment equations [Eqs. (3.12*b*)] state that for a body in equilibrium there is no resultant moment producing rotation about any axes parallel to any of the three coordinate axes.

Furthermore, in statics, a structure is usually considered rigid or nondeformable, since the forces acting on it cause very small deformations. It is assumed that no appreciable changes in dimensions occur because of applied loading. For some structures, however, such changes in dimensions may not be negligible. In these cases, the equations of equilibrium should be defined according to the deformed geometry of the structure (Art. 3.46).

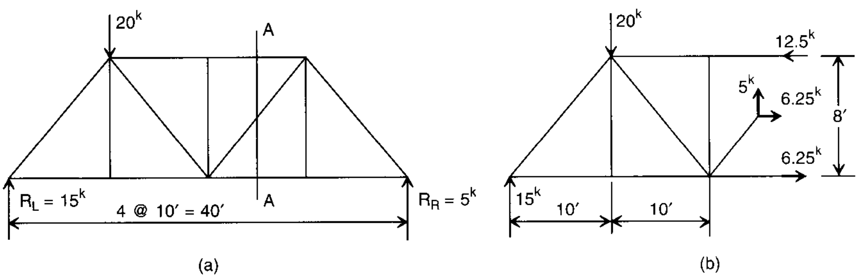


FIGURE 3.7 Forces acting on a truss. (*a*) Reactions R_L and R_R maintain equilibrium of the truss under 20-kip load. (*b*) Forces acting on truss members cut by section *A–A* maintain equilibrium.

(J. L. Meriam and L. G. Kraige, *Mechanics,* Part I: *Statics,* John Wiley & Sons, Inc., New York; F. P. Beer and E. R. Johnston, *Vector Mechanics for Engineers—Statics and Dynamics,* McGraw-Hill, Inc., New York.)

3.5 FRICTIONAL FORCES

Suppose a body A transmits a force $\mathbf{F}_{AB}$ onto a body B through a contact surface assumed to be flat (Fig. 3.8*a*). For the system to be in equilibrium, body B must react by applying an equal and opposite force $\mathbf{F}_{BA}$ on body A. $\mathbf{F}_{BA}$ may be resolved into a **normal force N** and a force $\mathbf{F}_f$ parallel to the plane of contact (Fig. 3.8*b*). The direction of $\mathbf{F}_f$ is drawn to resist motion.

The force $\mathbf{F}_f$ is called a **frictional force.** When there is no lubrication, the resistance to sliding is referred to as **dry friction.** The primary cause of dry friction is the microscopic roughness of the surfaces.

For a system including frictional forces to remain static (sliding not to occur), $\mathbf{F}_f$ cannot exceed a limiting value that depends partly on the normal force transmitted across the surface of contact. Because this limiting value also depends on the nature of the contact surfaces, it must be determined experimentally. For example, the limiting value is increased considerably if the contact surfaces are rough.

The limiting value of a frictional force for a body at rest is larger than the frictional force when sliding is in progress. The frictional force between two bodies that are motionless is called **static friction,** and the frictional force between two sliding surfaces is called **sliding** or **kinetic friction.**

Experiments indicate that the limiting force for dry friction $\mathbf{F}_u$ is proportional to the normal force $\mathbf{N}$:

$$\mathbf{F}_u = \mu_s \mathbf{N} \tag{3.13a}$$

where μ_s is the coefficient of static friction. For sliding not to occur, the frictional force $\mathbf{F}_f$ must be less than or equal to $\mathbf{F}_u$. If $\mathbf{F}_f$ exceeds this value, sliding will occur. In this case, the resulting frictional force is

$$\mathbf{F}_k = \mu_k \mathbf{N} \tag{3.13b}$$

where μ_k is the coefficient of kinetic friction.

Consider a block of negligible weight resting on a horizontal plane and subjected to a force $\mathbf{P}$ (Fig. 3.9*a*). From Eq. (3.1), the magnitudes of the components of $\mathbf{P}$ are

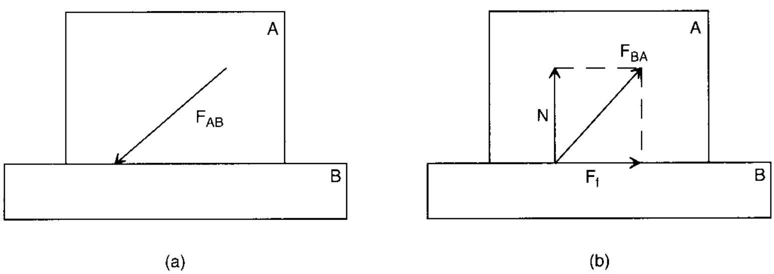


FIGURE 3.8 (*a*) Force $\mathbf{F}_{AB}$ tends to slide body A along the surface of body B. (*b*) Friction force $\mathbf{F}_f$ opposes motion.

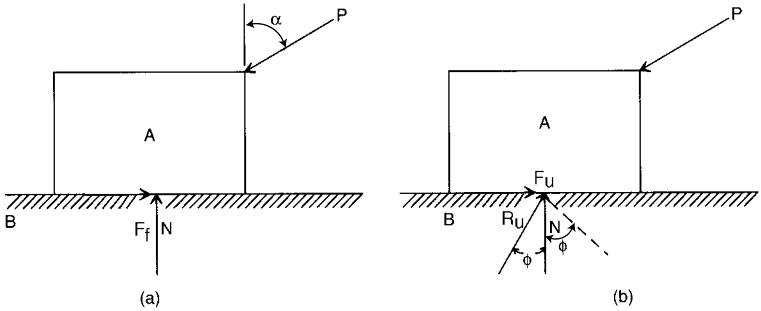


FIGURE 3.9 (*a*) Force **P** acting at an angle α tends to slide block *A* against friction with plane *B*. (*b*) When motion begins, the angle ϕ between the resultant **R** and the normal force **N** is the angle of static friction.

$$P_x = P \sin \alpha \tag{3.14a}$$

$$P_y = P \cos \alpha \tag{3.14b}$$

For the block to be in equilibrium, $\Sigma F_x = F_f - P_x = 0$ and $\Sigma F_y = N - P_y = 0$. Hence,

$$P_x = F_f \tag{3.15a}$$

$$P_y = N \tag{3.15b}$$

For sliding not to occur, the following inequality must be satisfied:

$$F_f \leq \mu_s N \tag{3.16}$$

Substitution of Eqs. (3.15) into Eq. (3.16) yields

$$P_x \leq \mu_s P_y \tag{3.17}$$

Substitution of Eqs. (3.14) into Eq. (3.17) gives

$$P \sin \alpha \leq \mu_s P \cos \alpha$$

which simplifies to

$$\tan \alpha \leq \mu_s \tag{3.18}$$

This indicates that the block will just begin to slide if the angle α is gradually increased to the angle of static friction ϕ, where $\tan \phi = \mu_s$ or $\phi = \tan^{-1} \mu_s$.

For the free-body diagram of the two-dimensional system shown in Fig. 3.9*b*, the resultant force $\mathbf{R}_u$ of forces $\mathbf{F}_u$ and **N** defines the bounds of a plane sector with angle 2ϕ. For motion not to occur, the resultant force **R** of forces $\mathbf{F}_f$ and **N** (Fig. 3.9*a*) must reside within this plane sector. In three-dimensional systems, no motion occurs when **R** is located within a cone of angle 2ϕ, called the **cone of friction.**

(F. P. Beer and E. R. Johnston, *Vector Mechanics for Engineers—Statics and Dynamics,* McGraw-Hill, Inc., New York.)

STRUCTURAL MECHANICS—DYNAMICS

Dynamics is that branch of mechanics which deals with bodies in motion. Dynamics is further divided into **kinematics,** the study of motion without regard to the forces causing the motion, and **kinetics,** the study of the relationship between forces and resulting motions.

3.6 KINEMATICS

Kinematics relates displacement, velocity, acceleration, and time. Most engineering problems in kinematics can be solved by assuming that the moving body is rigid and the motions occur in one plane.

Plane motion of a rigid body may be divided into four categories: **rectilinear translation,** in which all points of the rigid body move in straight lines; **curvilinear translation,** in which all points of the body move on congruent curves; **rotation,** in which all particles move in a circular path; and **plane motion,** a combination of translation and rotation in a plane.

Rectilinear translation is often of particular interest to designers. Let an arbitrary point P displace a distance Δs to P' during time interval Δt. The average velocity of the point during this interval is $\Delta s/\Delta t$. The **instantaneous velocity** is obtained by letting Δt approach zero:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \tag{3.19}$$

Let Δv be the difference between the instantaneous velocities at points P and P' during the time interval Δt. The average acceleration is $\Delta v/\Delta t$. The **instantaneous acceleration** is

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 s}{dt^2} \tag{3.20}$$

Suppose, for example, that the motion of a particle is described by the time-dependent displacement function $s(\mathrm{t}) = t^4 - 2t^2 + 1$. By Eq. (3.19), the velocity of the particle would be

$$v = \frac{ds}{dt} = 4t^3 - 4t$$

By Eq. (3.20), the acceleration of the particle would be

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = 12t^2 - 4$$

With the same relationships, the displacement function $s(t)$ could be determined from a given acceleration function $a(t)$. This can be done by integrating the acceleration function twice with respect to time t. The first integration would yield the velocity function $v(t) = \int a(t)\, dt$, and the second would yield the displacement function $s(t) = \int\int a(t)\, dt\, dt$.

These concepts can be extended to incorporate the relative motion of two points A and B in a plane. In general, the displacement $\mathbf{s}_A$ of A equals the vector sum of the displacement of $\mathbf{s}_B$ of B and the displacement $\mathbf{s}_{AB}$ of A relative to B:

$$\mathbf{s}_A = \mathbf{s}_B + \mathbf{s}_{AB} \tag{3.21}$$

Differentiation of Eq. (3.21) with respect to time gives the velocity relation

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{AB} \tag{3.22}$$

The acceleration of A is related to that of B by the vector sum

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{AB} \tag{3.23}$$

These equations hold for any two points in a plane. They need not be points on a rigid body.

(J. L. Meriam and L. G. Kraige, *Mechanics,* Part II: *Dynamics,* John Wiley & Son, Inc., New York; F. P. Beer and E. R. Johnston, *Vector Mechanics for Engineers—Statics and Dynamics,* McGraw-Hill, Inc., New York.)

3.7 KINETICS

Kinetics is that part of dynamics that includes the relationship between forces and any resulting motion.

Newton's second law relates force and acceleration by

$$\mathbf{F} = m\mathbf{a} \tag{3.24}$$

where the force **F** and the acceleration **a** are vectors having the same direction, and the mass m is a scalar.

The acceleration, for example, of a particle of mass m subject to the action of concurrent forces, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, can be determined from Eq. (3.24) by resolving each of the forces into three mutually perpendicular directions x, y, and z. The sums of the components in each direction are given by

$$\Sigma F_x = F_{1x} + F_{2x} + F_{3x} \tag{3.25a}$$

$$\Sigma F_y = F_{1y} + F_{2y} + F_{3y} \tag{3.25b}$$

$$\Sigma F_z = F_{1z} + F_{2z} + F_{3z} \tag{3.25c}$$

The magnitude of the resultant of the three concurrent forces is

$$\Sigma F = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2} \tag{3.26}$$

The acceleration of the particle is related to the force resultant by

$$\Sigma\mathbf{F} = m\mathbf{a} \tag{3.27}$$

The acceleration can then be determined from

$$\mathbf{a} = \frac{\Sigma\mathbf{F}}{m} \tag{3.28}$$

In a similar manner, the magnitudes of the components of the acceleration vector **a** are

$$a_x = \frac{d^2x}{dt^2} = \frac{\Sigma F_x}{m} \tag{3.29a}$$

$$a_y = \frac{d^2y}{dt^2} = \frac{\Sigma F_y}{m} \tag{3.29b}$$

$$a_z = \frac{d^2z}{dt^2} = \frac{\Sigma F_z}{m} \tag{3.29c}$$

Transformation of Eq. (3.27) into the form

$$\Sigma \mathbf{F} - m\mathbf{a} = 0 \tag{3.30}$$

provides a condition in dynamics that often can be treated as an instantaneous condition in statics; i.e., if a mass is suddenly accelerated in one direction by a force or a system of forces, an inertia force $m\mathbf{a}$ will be developed in the opposite direction so that the mass remains in a condition of dynamic equilibrium. This concept is known as **d'Alembert's principle.**

The principle of motion for a single particle can be extended to any number of particles in a system:

$$\Sigma F_x = \Sigma m_i a_{ix} = m\bar{a}_x \tag{3.31a}$$

$$\Sigma F_y = \Sigma m_i a_{iy} = m\bar{a}_y \tag{3.31b}$$

$$\Sigma F_z = \Sigma m_i a_{iz} = m\bar{a}_z \tag{3.31c}$$

where, for example, ΣF_x = algebraic sum of all x-component forces acting on the system of particles
$\Sigma m_i a_{ix}$ = algebraic sum of the products of the mass of each particle and the x component of its acceleration
m = total mass of the system
$\bar{a}_x$ = acceleration of the center of the mass of the particles in the x direction

Extension of these relationships permits calculation of the location of the center of mass (centroid for a homogeneous body) of an object:

$$\bar{x} = \frac{\Sigma m_i x_i}{m} \tag{3.32a}$$

$$\bar{y} = \frac{\Sigma m_i y_i}{m} \tag{3.32b}$$

$$\bar{z} = \frac{\Sigma m_i z_i}{m} \tag{3.32c}$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ = coordinates of center of mass of the system
m = total mass of the system
$\Sigma m_i x_i$ = algebraic sum of the products of the mass of each particle and its x coordinate
$\Sigma m_i y_i$ = algebraic sum of the products of the mass of each particle and its y coordinate
$\Sigma m_i z_i$ = algebraic sum of the products of the mass of each particle and its z coordinate

Concepts of impulse and momentum are useful in solving problems where forces are expressed as a function of time. These problems include both the kinematics and the kinetics parts of dynamics.

By Eqs. (3.29), the equations of motion of a particle with mass m are

$$\Sigma F_x = ma_x = m \frac{dv_x}{dt} \tag{3.33a}$$

$$\Sigma F_y = ma_y = m \frac{dv_y}{dt} \tag{3.33b}$$

$$\Sigma F_z = ma_z = m \frac{dv_z}{dt} \tag{3.33c}$$

Since m for a single particle is constant, these equations also can be written as

$$\Sigma F_x \, dt = d(mv_x) \tag{3.34a}$$

$$\Sigma F_y \, dt = d(mv_y) \tag{3.34b}$$

$$\Sigma F_z \, dt = d(mv_z) \tag{3.34c}$$

The product of mass and linear velocity is called **linear momentum.** The product of force and time is called **linear impulse.**

Equations (3.34) are an alternate way of stating Newton's second law. The action of ΣF_x, ΣF_y, and ΣF_z during a finite interval of time t can be found by integrating both sides of Eqs. (3.34):

$$\int_{t0}^{t1} \Sigma F_x \, dt = m(v_x)_{t1} - m(v_x)_{t0} \tag{3.35a}$$

$$\int_{t0}^{t1} \Sigma F_y \, dt = m(v_y)_{t1} - m(v_y)_{t0} \tag{3.35b}$$

$$\int_{t0}^{t1} \Sigma F_z \, dt = m(v_z)_{t1} - m(v_z)_{t0} \tag{3.35c}$$

That is, **the sum of the impulses on a body equals its change in momentum.**

(J. L. Meriam and L. G. Kraige, *Mechanics,* Part II: *Dynamics,* John Wiley & Sons, Inc., New York; F. P. Beer and E. R. Johnston, *Vector Mechanics for Engineers—Statics and Dynamics,* McGraw-Hill, Inc., New York.)

MECHANICS OF MATERIALS

Mechanics of materials, or **strength of materials,** incorporates the strength and stiffness properties of a material into the static and dynamic behavior of a structure.

3.8 STRESS-STRAIN DIAGRAMS

Suppose that a homogeneous steel bar with a constant cross-sectional area A is subjected to tension under axial load P (Fig. 3.10*a*). A gage length L is selected away from the ends of

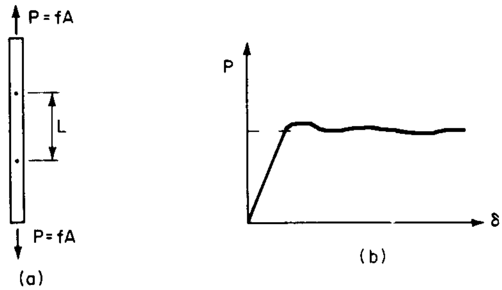


FIGURE 3.10 Elongations of test specimen (*a*) are measured from gage length *L* and plotted in (*b*) against load.

the bar, to avoid disturbances by the end attachments that apply the load. The load P is increased in increments, and the corresponding elongation δ of the original gage length is measured. Figure 3.10*b* shows the plot of a typical load-deformation relationship resulting from this type of test.

Assuming that the load is applied concentrically, the **strain** at any point along the gage length will be $\varepsilon = \delta/L$, and the **stress** at any point in the cross section of the bar will be $f = P/A$. Under these conditions, it is convenient to plot the relation between stress and strain. Figure 3.11 shows the resulting plot of a typical stress-stain relationship resulting from this test.

3.9 COMPONENTS OF STRESS AND STRAIN

Suppose that a plane cut is made through a solid in equilibrium under the action of some forces (Fig. 3.12*a*). The distribution of force on the area A in the plane may be represented by an equivalent resultant force $\mathbf{R_A}$ through point O (also in the plane) and a couple producing moment $\mathbf{M_A}$ (Fig. 3.12*b*).

Three mutually perpendicular axes x, y, and z at point O are chosen such that axis x is normal to the plane and y and z are in the plane. $\mathbf{R_A}$ can be resolved into components $\mathbf{R}_x$, $\mathbf{R}_y$, and $\mathbf{R}_z$, and $\mathbf{M_A}$ can be resolved into $\mathbf{M}_x$, $\mathbf{M}_y$, and $\mathbf{M}_z$ (Fig. 3.12*c*). Component $\mathbf{R}_x$ is called **normal force.** $\mathbf{R}_y$ and $\mathbf{R}_z$ are called **shearing forces.** Over area A, these forces produce an average **normal stress** R_x/A and average **shear stresses** R_y/A and R_z/A, respectively. If the area of interest is shrunk to an infinitesimally small area around point O, then the average stresses would approach limits, called **stress components,** f_x, v_{xy}, and v_{xz}, at point O. Thus, as indicated in Fig. 3.12*d*,

$$f_x = \lim_{A \to 0} \left(\frac{R_x}{A}\right) \tag{3.36a}$$

$$v_{xy} = \lim_{A \to 0} \left(\frac{R_y}{A}\right) \tag{3.36b}$$

$$v_{xz} = \lim_{A \to 0} \left(\frac{R_z}{A}\right) \tag{3.36c}$$

Because the moment $\mathbf{M_A}$ and its corresponding components are all taken about point O, they are not producing any additional stress at this point.

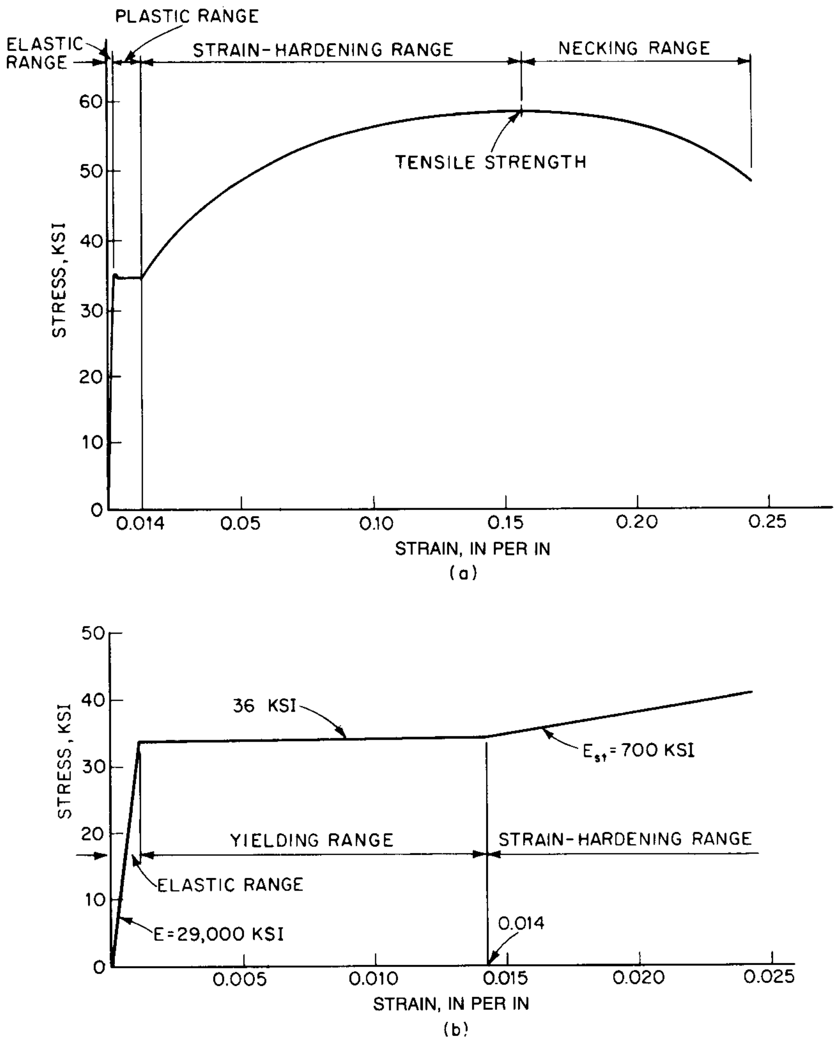


FIGURE 3.11 (*a*) Stress-strain diagram for A36 steel. (*b*) Portion of that diagram in the yielding range.

If another plane is cut through O that is normal to the y axis, the area surrounding O in this plane will be subjected to a different resultant force and moment through O. If the area is made to approach zero, the stress components f_y, v_{yx}, and v_{yz} are obtained. Similarly, if a third plane cut is made through O, normal to the z direction, the stress components are f_z, v_{zx}, v_{zy}.

The normal-stress component is denoted by f and a single subscript, which indicates the direction of the axis normal to the plane. The shear-stress component is denoted by v and two subscripts. The first subscript indicates the direction of the normal to the plane, and the second subscript indicates the direction of the axis to which the component is parallel.

The state of stress at a point O is shown in Fig. 3.13 on a rectangular parallelepiped with length of sides Δx, Δy, and Δx. The parallelepiped is taken so small that the stresses can be

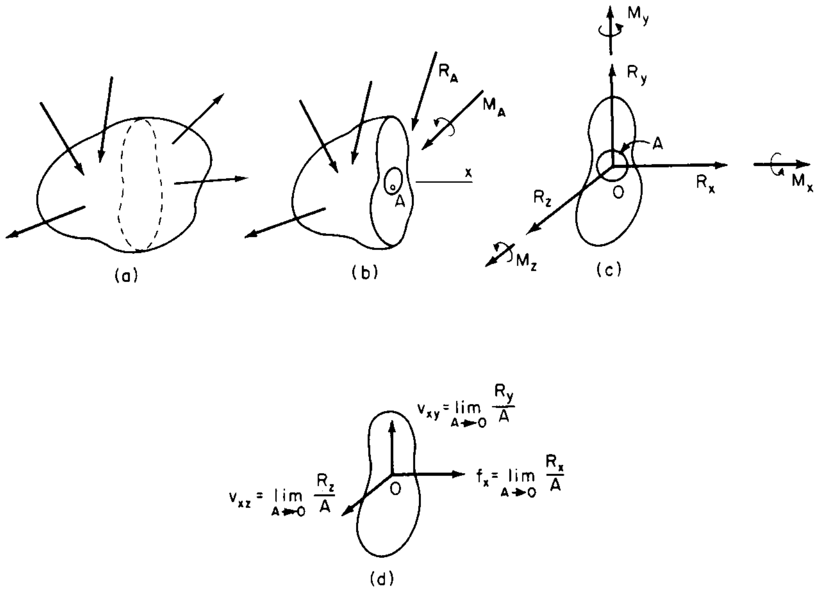


FIGURE 3.12 Stresses at a point in a body due to external loads. (*a*) Forces acting on the body. (*b*) Forces acting on a portion of the body. (*c*) Resolution of forces and moments about coordinate axes through point *O*. (*d*) Stresses at point *O*.

considered uniform and equal on parallel faces. The stress at the point can be expressed by the nine components shown. Some of these components, however, are related by equilibrium conditions:

$$v_{xy} = v_{yx} \qquad v_{yz} = v_{zy} \qquad v_{zx} = v_{xz} \tag{3.37}$$

Therefore, the actual state of stress has only six independent components.

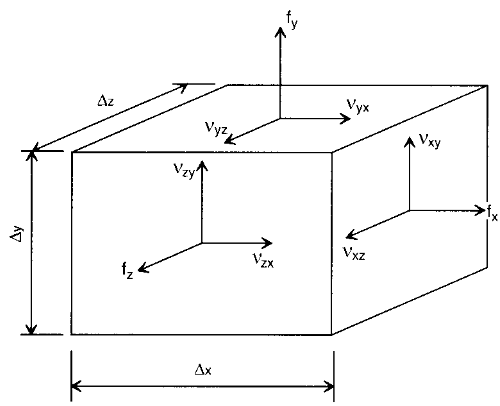


FIGURE 3.13 Components of stress at a point.

A component of strain corresponds to each component of stress. Normal strains ε_x, ε_y, and ε_z are the changes in unit length in the x, y, and z directions, respectively, when the deformations are small (for example, ϵ_x is shown in Fig. 3.14*a*). Shear strains γ_{xy}, γ_{zy}, and γ_{zx} are the decreases in the right angle between lines in the body at O parallel to the x and y, z and y, and z and x axes, respectively (for example, γ_{xy} is shown in Fig. 3.14*b*). Thus, similar to a state of stress, a state of strain has nine components, of which six are independent.

3.10 STRESS-STRAIN RELATIONSHIPS

Structural steels display linearly elastic properties when the load does not exceed a certain limit. Steels also are **isotropic;** i.e., the elastic properties are the same in all directions. The material also may be assumed **homogeneous,** so the smallest element of a steel member possesses the same physical property as the member. It is because of these properties that there is a linear relationship between components of stress and strain. Established experimentally (see Art. 3.8), this relationship is known as **Hooke's law.** For example, in a bar subjected to axial load, the normal strain in the axial direction is proportional to the normal stress in that direction, or

$$\varepsilon = \frac{f}{E} \tag{3.38}$$

where E is the **modulus of elasticity,** or **Young's modulus.**

If a steel bar is stretched, the width of the bar will be reduced to account for the increase in length (Fig. 3.14*a*). Thus the normal strain in the x direction is accompanied by lateral strains of opposite sign. If ϵ_x is a tensile strain, for example, the lateral strains in the y and z directions are contractions. These strains are related to the normal strain and, in turn, to the normal stress by

$$\varepsilon_y = -\nu\varepsilon_x = -\nu\frac{f_x}{E} \tag{3.39a}$$

$$\varepsilon_z = -\nu\varepsilon_x = -\nu\frac{f_x}{E} \tag{3.39b}$$

where ν is a constant called **Poisson's ratio.**

If an element is subjected to the action of simultaneous normal stresses f_x, f_y, and f_z uniformly distributed over its sides, the corresponding strains in the three directions are

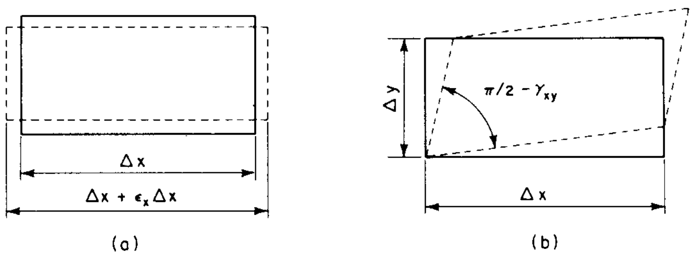


FIGURE 3.14 (*a*) Normal deformation. (*b*) Shear deformation.

$$\varepsilon_x = \frac{1}{E}[f_x - \nu(f_y + f_z)] \tag{3.40a}$$

$$\varepsilon_y = \frac{1}{E}[f_y - \nu(f_x + f_z)] \tag{3.40b}$$

$$\varepsilon_z = \frac{1}{E}[f_z - \nu(f_x + f_y)] \tag{3.40c}$$

Similarly, shear strain γ is linearly proportional to shear stress v

$$\gamma_{xy} = \frac{v_{xy}}{G} \qquad \gamma_{yz} = \frac{v_{yz}}{G} \qquad \gamma_{zx} = \frac{v_{zx}}{G} \tag{3.41}$$

where the constant G is the **shear modulus of elasticity,** or **modulus of rigidity.** For an isotropic material such as steel, G is directly proportional to E:

$$G = \frac{E}{2(1 + \nu)} \tag{3.42}$$

The analysis of many structures is simplified if the stresses are parallel to one plane. In some cases, such as a thin plate subject to forces along its edges that are parallel to its plane and uniformly distributed over its thickness, the stress distribution occurs all in one plane. In this case of **plane stress,** one normal stress, say f_z, is zero, and corresponding shear stresses are zero: $v_{zx} = 0$ and $v_{zy} = 0$.

In a similar manner, if all deformations or strains occur within a plane, this is a condition of **plane strain.** For example, $\varepsilon_z = 0$, $\gamma_{zx} = 0$, and $\gamma_{zy} = 0$.

3.11 PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

When stress components relative to a defined set of axes are given at any point in a condition of plane stress or plane strain (see Art. 3.10), this state of stress may be expressed with respect to a different set of axes that lie in the same plane. For example, the state of stress at point O in Fig. 3.15*a* may be expressed in terms of either the x and y axes with stress components, f_x, f_y, and v_{xy} or the x' and y' axes with stress components $f_{x'}$, $f_{y'}$, and $v_{x'y'}$ (Fig. 3.15*b*). If stress components f_x, f_y, and v_{xy} are given and the two orthogonal coordinate systems differ by an angle α with respect to the original x axis, the stress components $f_{x'}$, $f_{y'}$, and $v_{x'y'}$ can be determined by statics. The transformation equations for stress are

$$f_{x'} = \tfrac{1}{2}(f_x + f_y) + \tfrac{1}{2}(f_x - f_y)\cos 2\alpha + v_{xy}\sin 2\alpha \tag{3.43a}$$

$$f_{y'} = \tfrac{1}{2}(f_x + f_y) - \tfrac{1}{2}(f_x - f_y)\cos 2\alpha - v_{xy}\sin 2\alpha \tag{3.43b}$$

$$v_{x'y'} = -\tfrac{1}{2}(f_x - f_y)\sin 2\alpha + v_{xy}\cos 2\alpha \tag{3.43c}$$

From these equations, an angle α_p can be chosen to make the shear stress $v_{x'y'}$ equal zero. From Eq. (3.43*c*), with $v_{x'y'} = 0$,

$$\tan 2\alpha_p = \frac{2v_{xy}}{f_x - f_y} \tag{3.44}$$

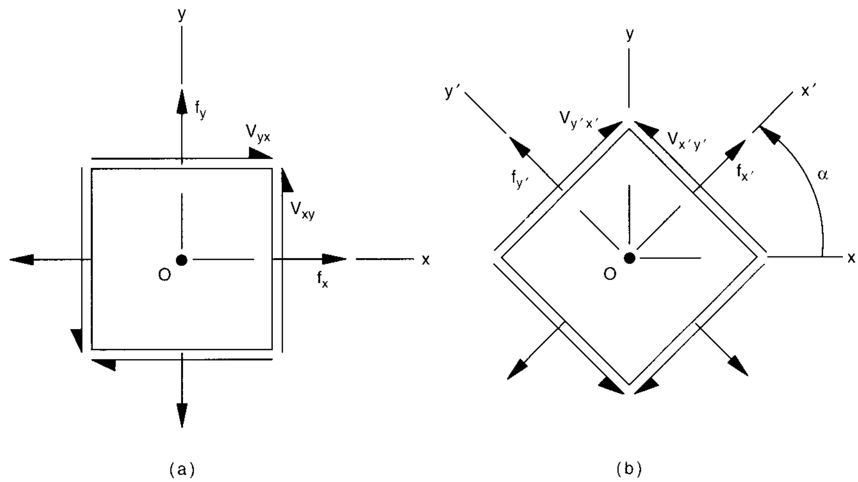


FIGURE 3.15 (*a*) Stresses at point *O* on planes perpendicular to *x* and *y* axes. (*b*) Stresses relative to rotated axes.

This equation indicates that two perpendicular directions, α_p and $\alpha_p + (\pi/2)$, may be found for which the shear stress is zero. These are called **principal directions.** On the plane for which the shear stress is zero, one of the normal stresses is the maximum stress f_1 and the other is the minimum stress f_2 for all possible states of stress at that point. Hence the normal stresses on the planes in these directions are called the **principal stresses.** The magnitude of the principal stresses may be determined from

$$f = \frac{f_x + f_y}{2} \pm \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + {v_{xy}}^2} \tag{3.45}$$

where the algebraically larger principal stress is given by f_1 and the minimum principal stress is given by f_2.

Suppose that the x and y directions are taken as the principal directions, that is, $v_{xy} = 0$. Then Eqs. (3.43) may be simplified to

$$f_{x'} = \tfrac{1}{2}(f_1 + f_2) + \tfrac{1}{2}(f_1 - f_2)\cos 2\alpha \tag{3.46a}$$

$$f_{y'} = \tfrac{1}{2}(f_1 + f_2) - \tfrac{1}{2}(f_1 - f_2)\cos 2\alpha \tag{3.46b}$$

$$v_{x'y'} = -\tfrac{1}{2}(f_1 - f_2)\sin 2\alpha \tag{3.46c}$$

By Eq. (3.46*c*), the maximum shear stress occurs when $\sin 2\alpha = \pi/2$, i.e., when $\alpha = 45°$. Hence the maximum shear stress occurs on each of two planes that bisect the angles between the planes on which the principal stresses act. The magnitude of the maximum shear stress equals one-half the algebraic difference of the principal stresses:

$$v_{max} = -\tfrac{1}{2}(f_1 - f_2) \tag{3.47}$$

If on any two perpendicular planes through a point only shear stresses act, the state of stress at this point is called **pure shear.** In this case, the principal directions bisect the angles between the planes on which these shear stresses occur. The principal stresses are equal in magnitude to the unit shear stress in each plane on which only shears act.

3.12 MOHR'S CIRCLE

Equations (3.46) for stresses at a point O can be represented conveniently by **Mohr's circle** (Fig. 3.16). Normal stress f is taken as the abscissa, and shear stress v is taken as the ordinate. The center of the circle is located on the f axis at $(f_1 + f_2)/2$, where f_1 and f_2 are the maximum and minimum principal stresses at the point, respectively. The circle has a radius of $(f_1 - f_2)/2$. For each plane passing through the point O there are two diametrically opposite points on Mohr's circle that correspond to the normal and shear stresses on the plane. Thus Mohr's circle can be used conveniently to find the normal and shear stresses on a plane when the magnitude and direction of the principal stresses at a point are known.

Use of Mohr's circle requires the principal stresses f_1 and f_2 to be marked off on the abscissa (points A and B in Fig. 3.16, respectively). Tensile stresses are plotted to the right of the v axis and compressive stresses to the left. (In Fig. 3.16, the principal stresses are indicated as tensile stresses.) A circle is then constructed that has radius $(f_1 + f_2)/2$ and passes through A and B. The normal and shear stresses f_x, f_y, and v_{xy} on a plane at an angle α with the principal directions are the coordinates of points C and D on the intersection of

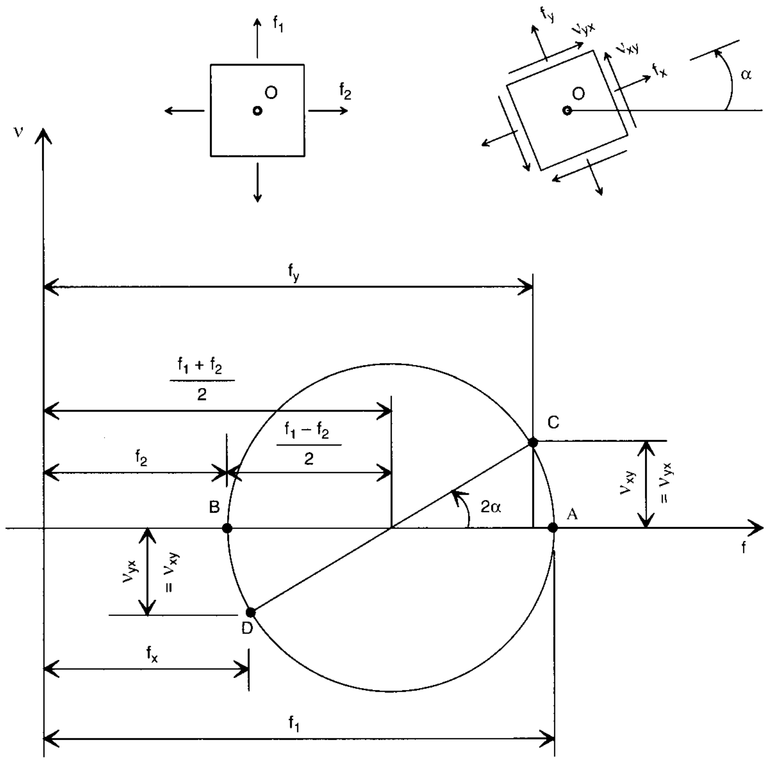


FIGURE 3.16 Mohr circle for obtaining, from principal stresses at a point, shear and normal stresses on any plane through the point.

the circle and the diameter making an angle 2α with the abscissa. A counterclockwise angle change α in the stress plane represents a counterclockwise angle change of 2α on Mohr's circle. The stresses f_x, v_{xy}, and f_y, v_{yx} on two perpendicular planes are represented on Mohr's circle by points (f_x, $-v_{xy}$) and (f_y, v_{yx}), respectively. Note that a shear stress is defined as positive when it tends to produce counter-clockwise rotation of the element.

Mohr's circle also can be used to obtain the principal stresses when the normal stresses on two perpendicular planes and the shearing stresses are known. Figure 3.17 shows construction of Mohr's circle from these conditions. Points C (f_x, v_{xy}) and D (f_y, $-v_{xy}$) are plotted and a circle is constructed with CD as a diameter. Based on this geometry, the abscissas of points A and B that correspond to the principal stresses can be determined.

(I. S. Sokolnikoff, *Mathematical Theory of Elasticity;* S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity;* and Chi-Teh Wang, *Applied Elasticity;* and F. P. Beer and E. R. Johnston, *Mechanics of Materials,* McGraw-Hill, Inc., New York; A. C. Ugural and S. K. Fenster, *Advanced Strength and Applied Elasticity,* Elsevier Science Publishing, New York.)

BASIC BEHAVIOR OF STRUCTURAL COMPONENTS

The combination of the concepts for statics (Arts 3.2 to 3.5) with those of mechanics of materials (Arts. 3.8 to 3.12) provides the essentials for predicting the basic behavior of members in a structural system.

Structural members often behave in a complicated and uncertain way. To analyze the behavior of these members, i.e., to determine the relationships between the external loads and the resulting internal stresses and deformations, certain idealizations are necessary. Through this approach, structural members are converted to such a form that an analysis of their behavior in service becomes readily possible. These idealizations include mathematical models that represent the type of structural members being assumed and the structural support conditions (Fig. 3.18).

3.13 TYPES OF STRUCTURAL MEMBERS AND SUPPORTS

Structural members are usually classified according to the principal stresses induced by loads that the members are intended to support. **Axial-force members** (**ties** or **struts**) are those subjected to only tension or compression. A **column** is a member that may buckle under compressive loads due to its slenderness. **Torsion members,** or **shafts,** are those subjected to twisting moment, or torque. A **beam** supports loads that produce bending moments. A **beam-column** is a member in which both bending moment and compression are present.

In practice, it may not be possible to erect truly axially loaded members. Even if it were possible to apply the load at the centroid of a section, slight irregularities of the member may introduce some bending. For analysis purposes, however, these bending moments may often be ignored, and the member may be idealized as axially loaded.

There are three types of ideal supports (Fig. 3.19). In most practical situations, the support conditions of structures may be described by one of these three. Figure 3.19*a* represents a support at which horizontal movement and rotation are unrestricted, but vertical movement is restrained. This type of support is usually shown by **rollers.** Figure 3.19*b* represents a **hinged,** or **pinned support,** at which vertical and horizontal movements are prevented, while only rotation is permitted. Figure 3.19*c* indicates a **fixed support,** at which no translation or rotation is possible.

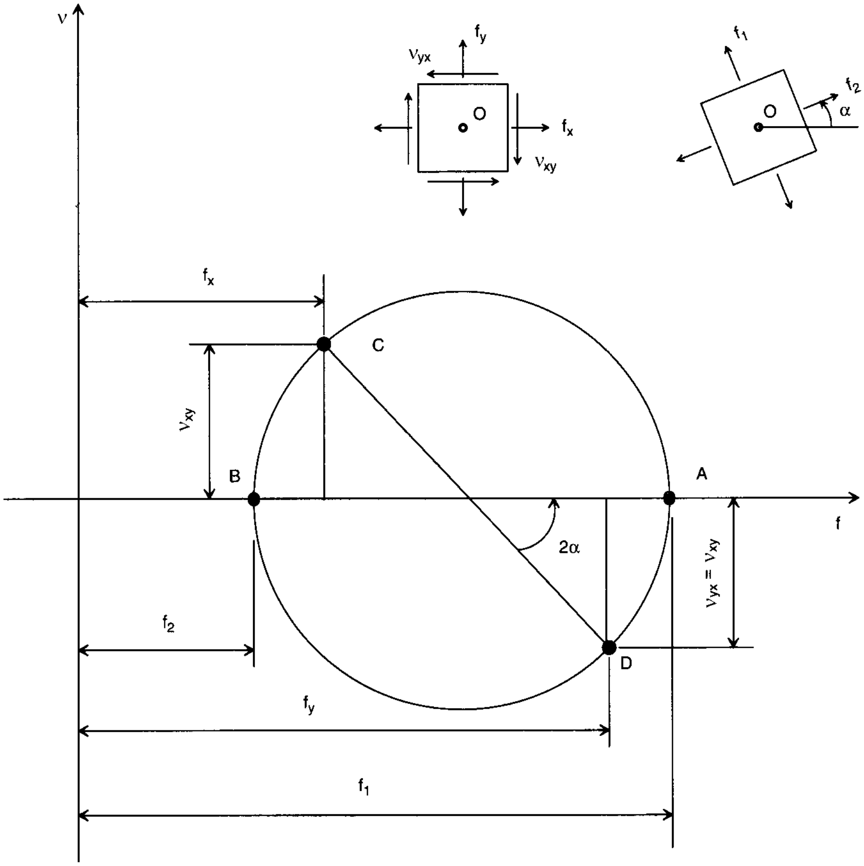


FIGURE 3.17 Mohr circle for determining principal stresses at a point.

3.14 AXIAL-FORCE MEMBERS

In an axial-force member, the stresses and strains are uniformly distributed over the cross section. Typically examples of this type of member are shown in Fig. 3.20.

Since the stress is constant across the section, the equation of equilibrium may be written as

$$P = Af \tag{3.48}$$

where P = axial load
f = tensile, compressive, or bearing stress
A = cross-sectional area of the member

Similarly, if the strain is constant across the section, the strain ε corresponding to an axial tensile or compressive load is given by

$$\varepsilon = \frac{\Delta}{L} \tag{3.49}$$

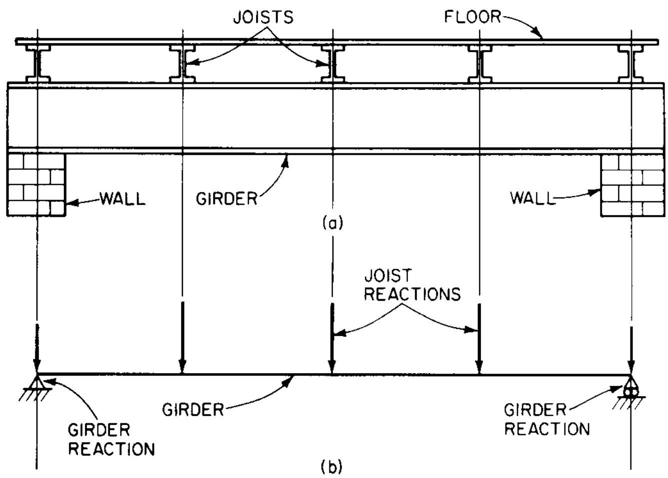


FIGURE 3.18 Idealization of (*a*) joist-and-girder framing by (*b*) concentrated loads on a simple beam.

where L = length of member
Δ = change in length of member

Assuming that the material is an isotropic linear elastic medium (see Art. 3.9), Eqs. (3.48) and (3.49) are related according to Hooke's law $\varepsilon = f/E$, where E is the modulus of elasticity of the material. The change in length Δ of a member subjected to an axial load P can then be expressed by

$$\Delta = \frac{PL}{AE} \tag{3.50}$$

Equation (3.50) relates the load applied at the ends of a member to the displacement of one end of the member relative to the other end. The factor L/AE represents the **flexibility** of the member. It gives the displacement due to a unit load.

Solving Eq. (3.50) for P yields

$$P = \frac{AE}{L}\Delta \tag{3.51}$$

The factor AE/L represents the **stiffness** of the member in resisting axial loads. It gives the magnitude of an axial load needed to produce a unit displacement.

Equations (3.50) to (3.51) hold for both tension and compression members. However, since compression members may buckle prematurely, these equations may apply only if the member is relatively short (Arts. 3.46 and 3.49).

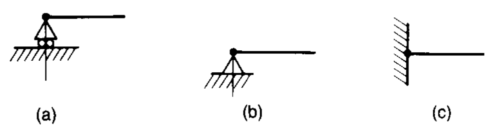


FIGURE 3.19 Representation of types of ideal supports: (*a*) roller, (*b*) hinged support, (*c*) fixed support.

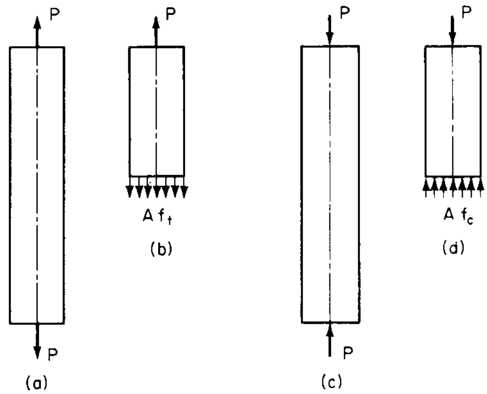


FIGURE 3.20 Stresses in axially loaded members: (*a*) bar in tension, (*b*) tensile stresses in bar, (*c*) strut in compression, (*d*) compressive stresses in strut.

3.15 MEMBERS SUBJECTED TO TORSION

Forces or moments that tend to twist a member are called **torisonal loads.** In shafts, the stresses and corresponding strains induced by these loads depend on both the shape and size of the cross section.

Suppose that a circular shaft is fixed at one end and a twisting couple, or **torque,** is applied at the other end (Fig. 3.21*a*). When the angle of twist is small, the circular cross section remains circular during twist. Also, the distance between any two sections remains the same, indicating that there is no longitudinal stress along the length of the member.

Figure 3.21*b* shows a cylindrical section with length *dx* isolated from the shaft. The lower cross section has rotated with respect to its top section through an angle $d\theta$, where θ is the

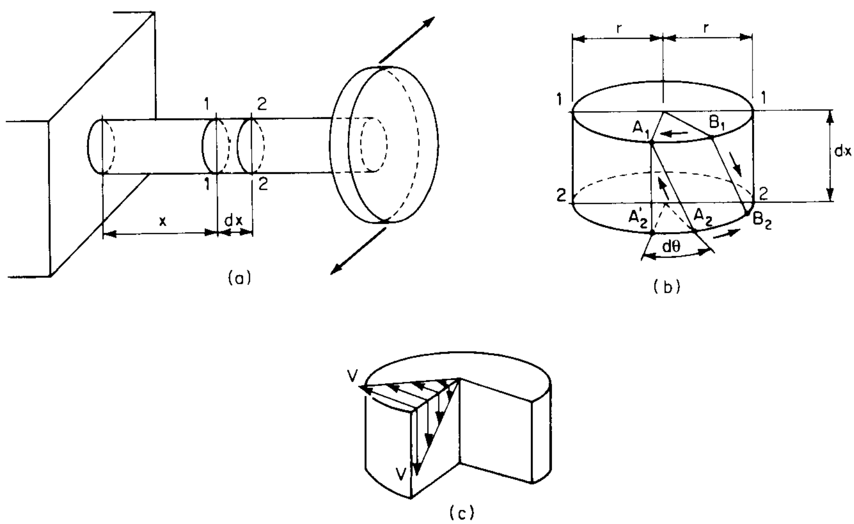


FIGURE 3.21 (*a*) Circular shaft in torsion. (*b*) Deformation of a portion of the shaft. (*c*) Shear in shaft.

total rotation of the shaft with respect to the fixed end. With no stress normal to the cross section, the section is in a state of pure shear (Art. 3.9). The shear stresses act normal to the radii of the section. The magnitude of the shear strain γ at a given radius r is given by

$$\gamma = \frac{A_2A_2'}{A_1A_2'} = r\,\frac{d\theta}{dx} = \frac{r\theta}{L} \tag{3.52}$$

where L = total length of the shaft
$d\theta/dx = \theta/L$ = angle of twist per unit length of shaft

Incorporation of Hooke's law ($v = G\gamma$) into Eq. (3.52) gives the shear stress at a given radius r:

$$v = \frac{Gr\theta}{L} \tag{3.53}$$

where G is the shear modulus of elasticity. This equation indicates that the shear stress in a **circular** shaft varies directly with distance r from the axis of the shaft (Fig. 3.21*c*). The maximum shear stress occurs at the surface of the shaft.

From conditions of equilibrium, the twisting moment T and the shear stress v are related by

$$v = \frac{rT}{J} \tag{3.54}$$

where $J = \int r^2\,dA = \pi r^4/2$ = **polar moment of inertia**
dA = differential area of the circular section

By Eqs. (3.53) and (3.54), the applied torque T is related to the relative rotation of one end of the member to the other end by

$$T = \frac{GJ}{L}\,\theta \tag{3.55}$$

The factor GJ/L represents the stiffness of the member in resisting twisting loads. It gives the magnitude of a torque needed to produce a unit rotation.

Noncircular shafts behave differently under torsion from the way circular shafts do. In noncircular shafts, cross sections do not remain plane, and radial lines through the centroid do not remain straight. Hence the direction of the shear stress is not normal to the radius, and the distribution of shear stress is not linear. If the end sections of the shaft are free to warp, however, Eq. (3.55) may be applied generally when relating an applied torque T to the corresponding member deformation θ. Table 3.1 lists values of J and maximum shear stress for various types of sections.

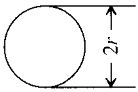
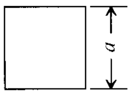
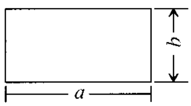
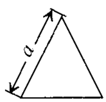
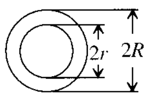
(*Torsional Analysis of Steel Members,* American Institute of Steel Construction; F. Arbabi, *Structural Analysis and Behavior,* McGraw-Hill, Inc., New York.)

3.16 BENDING STRESSES AND STRAINS IN BEAMS

Beams are structural members subjected to lateral forces that cause bending. There are distinct relationships between the load on a beam, the resulting internal forces and moments, and the corresponding deformations.

Consider the uniformly loaded beam with a symmetrical cross section in Fig. 3.22. Subjected to bending, the beam carries this load to the two supporting ends, one of which is hinged and the other of which is on rollers. Experiments have shown that strains developed

TABLE 3.1 Torsional Constants and Shears

	Polar moment of inertia J	Maximum shear* v_{max}
	$\frac{1}{2}\pi r^4$	$\frac{2T}{\pi r^3}$ at periphery
	$0.141a^4$	$\frac{T}{208a^3}$ at midpoint of each side
	$ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$	$\frac{T(3a + 1.8b)}{a^2b^2}$ at midpoint of longer sides
	$0.0217a^4$	$\frac{20T}{a^3}$ at midpoint of each side
	$\frac{1}{2}\pi(R^4 - r^4)$	$\frac{2TR}{\pi(R^4 - r^4)}$ at outer periphery

* T = twisting moment, or torque.

along the depth of a cross section of the beam vary linearly; i.e., a plane section before loading remains plane after loading. Based on this observation, the stresses at various points in a beam may be calculated if the stress-strain diagram for the beam material is known. From these stresses, the resulting internal forces at a cross section may be obtained.

Figure 3.23*a* shows the symmetrical cross section of the beam shown in Fig. 3.22. The strain varies linearly along the beam depth (Fig. 3.23*b*). The strain at the top of the section is compressive and decreases with depth, becoming zero at a certain distance below the top. The plane where the strain is zero is called the **neutral axis**. Below the neutral axis, tensile strains act, increasing in magnitude downward. With use of the stress-strain relationship of the material (e.g., see Fig. 3.11), the cross-sectional stresses may be computed from the strains (Fig. 3.23*c*).

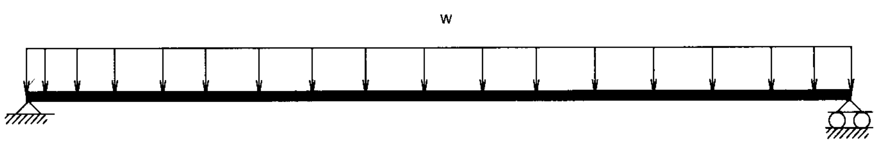


FIGURE 3.22 Uniformly loaded, simply supported beam.

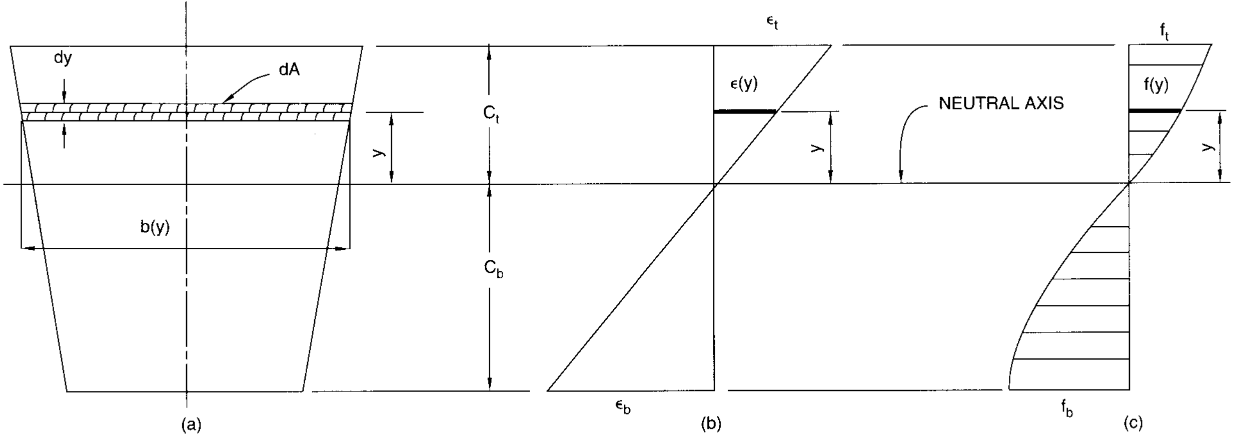


FIGURE 3.23 (*a*) Symmetrical section of a beam develops (*b*) linear strain distribution and (*c*) nonlinear stress distribution.

If the entire beam is in equilibrium, then all its sections also must be in equilibrium. With no external horizontal forces applied to the beam, the net internal horizontal forces any section must sum to zero:

$$\int_{c_b}^{c_t} f(y)\, dA = \int_{c_b}^{c_t} f(y)b(y)\, dy = 0 \tag{3.56}$$

where dA = differential unit of cross-sectional area located at a distance y from the neutral axis
$b(y)$ = width of beam at distance y from the neutral axis
$f(y)$ = normal stress at a distance y from the neutral axis
c_b = distance from neutral axis to beam bottom
c_t = distance from neutral axis to beam top

The moment M at this section due to internal forces may be computed from the stresses $f(y)$:

$$M = \int_{c_b}^{c_t} f(y)b(y)y\, dy \tag{3.57}$$

The moment M is usually considered positive when bending causes the bottom of the beam to be in tension and the top in compression. To satisfy equilibrium requirements, M must be equal in magnitude but opposite in direction to the moment at the section due to the loading.

3.16.1 Bending in the Elastic Range

If the stress-strain diagram is linear, the stresses would be linearly distributed along the depth of the beam corresponding to the linear distribution of strains:

$$f(y) = \frac{f_t}{c_t} y \tag{3.58}$$

where f_t = stress at top of beam
y = distance from the neutral axis

Substitution of Eq. (3.58) into Eq. (3.56) yields

$$\int_{c_b}^{c_t} \frac{f_t}{c_t} yb(y)\, dy = \frac{f_t}{c_t} \int_{c_b}^{c_t} yb(y)\, dy = 0 \tag{3.59}$$

Equation (3.59) provides a relationship that can be used to locate the neutral axis of the section. For the section shown in Fig. 3.23, Eq. (3.59) indicates that the neutral axis coincides with the centroidal axis.

Substitution of Eq. (3.58) into Eq. (3.57) gives

$$M = \int_{c_b}^{c_t} \frac{f_t}{c_t} b(y)y^2\, dy = \frac{f_t}{c_t} \int_{c_b}^{c_t} b(y)y^2\, dy = f_t \frac{I}{c_t} \tag{3.60}$$

where $\int_{c_b}^{c_t} b(y)y^2\, dy = I$ = **moment of inertia** of the cross section about the neutral axis. The factor I/c_t is the **section modulus** S_t for the top surface.

Substitution of f_t/c_t from Eq. (3.58) into Eq. (3.60) gives the relation between moment and stress at any distance y from the neutral axis:

$$M = \frac{I}{y} f(y) \tag{3.61a}$$

$$f(y) = M \frac{y}{I} \tag{3.61b}$$

Hence, for the bottom of the beam,

$$M = f_b \frac{I}{c_b} \tag{3.62}$$

where I/c_b is the section modulus S_b for the bottom surface.

For a section symmetrical about the neutral axis,

$$c_t = c_b \qquad f_t = f_b \qquad S_t = S_b \tag{3.63}$$

For example, a rectangular section with width b and depth d would have a moment of inertia $I = bd^3/12$ and a section modulus for both compression and tension $S = I/c = bd^2/6$. Hence,

$$M = Sf = \frac{bd^2}{6} f \tag{3.64a}$$

$$f = \frac{M}{S} = M \frac{6}{bd^2} \tag{3.64b}$$

The geometric properties of several common types of cross sections are given in Table 3.2

3.16.2 Bending in the Plastic Range

If a beam is heavily loaded, all the material at a cross section may reach the yield stress f_y [that is, $f(y) = \pm f_y$]. Although the strains would still vary linearly with depth (Fig. 3.24*b*), the stress distribution would take the form shown in Fig. 3.24*c*. In this case, Eq. (3.57) becomes the **plastic moment:**

$$M_p = f_y \int_0^{c_t} b(y)y\,dy + f_y \int_0^{c_b} b(y)y\,dy = Zf_y \tag{3.65}$$

where $\int_0^{c_t} b(y)y\,dy + f_y \int_0^{c_b} b(y)y\,dy = Z =$ **plastic section modulus.** For a rectangular section (Fig. 3.24*a*),

$$M_p = bf_y \int_0^{h/2} y\,dy + bf_y \int_0^{-h/2} y\,dy = \frac{bh^2}{4} f_y \tag{3.66}$$

Hence the plastic modulus Z equals $bh^2/4$ for a rectangular section.

3.17 SHEAR STRESSES IN BEAMS

In addition to normal stresses (Art. 3.16), beams are subjected to shearing. Shear stresses vary over the cross section of a beam. At every point in the section, there are both a vertical and a horizontal shear stress, equal in magnitude [Eq. (3.37)].

TABLE 3.2 Properties of Sections

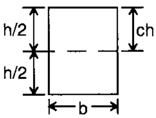
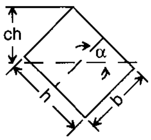
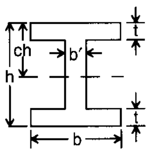
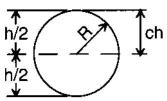
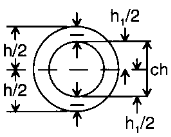
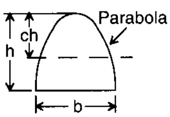
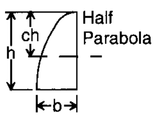
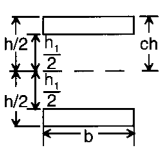
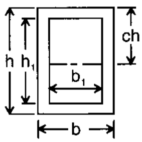
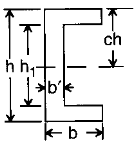
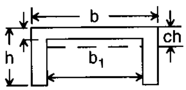
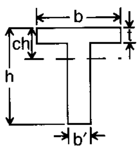
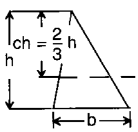
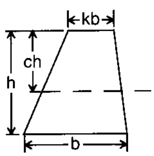
	$A = \dfrac{\text{Area}}{bh}$	c = depth to centroid ÷ h	I = moment of inertia about centroidal axis ÷ bh^3
	1.0	$\frac{1}{2}$	$\frac{1}{12}$
	1.0	$\frac{b}{2h}\sin\alpha + \frac{1}{2}\cos\alpha$	$\frac{1}{12}\left(\frac{b}{h}\sin\alpha\right)^2 + \frac{1}{12}\cos^2\alpha$
	$1 - \left(1 - \frac{b'}{b}\right)\left(1 - \frac{2t}{h}\right)$	$\frac{1}{2}$	$\frac{1}{12}\left[1 - \left(1 - \frac{b'}{b}\right)\left(1 - \frac{2t}{h}\right)^3\right]$
	$\frac{\pi}{4} = 0.785398$	$\frac{1}{2}$	$\frac{\pi}{64} = 0.049087$
	$\frac{\pi}{4}\left(1 - \frac{h_1^2}{h^2}\right)$	$\frac{1}{2}$	$\frac{\pi}{64}\left(1 - \frac{h_1^4}{h^4}\right)$
	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{8}{175}$
	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{8}{175}$
	$1 - \frac{h_1}{h}$	$\frac{1}{2}$	$\frac{1}{12}\left(1 - \frac{h_1^3}{h^3}\right)$

TABLE 3.2 Properties of Sections (*Continued*)

	$A = \frac{\text{Area}}{bh}$	c = depth to centroid ÷ h	I = moment of inertia about centroidal axis ÷ bh^3
	$1 - \frac{b_1}{b}\left(\frac{h_1}{h}\right)$	$\frac{1}{2}$	$\frac{1}{12}\left[1 - \frac{b_1}{b}\left(\frac{h_1}{h}\right)^3\right]$
	$1 - \left(1 - \frac{b'}{b}\right)\left(\frac{h_1}{h}\right)$	$\frac{1}{2}$	$\frac{1}{12}\left[1 - \left(1 - \frac{b'}{b}\right)\left(\frac{h_1}{h}\right)^3\right]$
	$1 - \frac{b_1}{b}\left(1 - \frac{t}{h}\right)$	$\frac{1}{2}\,\frac{1 - \frac{b_1}{b}\left(1 - \frac{t^2}{h^2}\right)}{1 - \frac{b_1}{b}\left(1 - \frac{t}{h}\right)}$	$\frac{1}{3}\left\{1 - a\left(1 - \frac{t^3}{h^3}\right) - \frac{3}{4}\,\frac{\left[1 - a\left(1 - \frac{t^2}{h^2}\right)\right]^2}{\left[1 - a\left(1 - \frac{t}{h}\right)\right]}\right\}$ $a = \frac{b_1}{b}$
	$\frac{t}{h} + \frac{b'}{b}\left(1 - \frac{t}{h}\right)$	$\frac{1}{2}\,\frac{\left(\frac{t}{h}\right)^2 + \frac{b'}{b}\left(1 - \frac{t^2}{h^2}\right)}{\frac{t}{h} + \frac{b'}{b}\left(1 - \frac{t}{h}\right)}$	$\frac{1}{3}\left\{1 - a\left(1 - \frac{t^3}{h^3}\right) - \frac{3}{4}\,\frac{\left[1 - a\left(1 - \frac{t^2}{h^2}\right)\right]^2}{\left[1 - a\left(1 - \frac{t}{h}\right)\right]}\right\}$ $a = \frac{b - b'}{b}$
	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{36}$
	$\frac{(1 + k)}{2}$	$\frac{(2 + k)}{3(1 + k)}$	$\frac{1}{36}\,\frac{(1 + 4k + k^2)}{(1 + k)}$

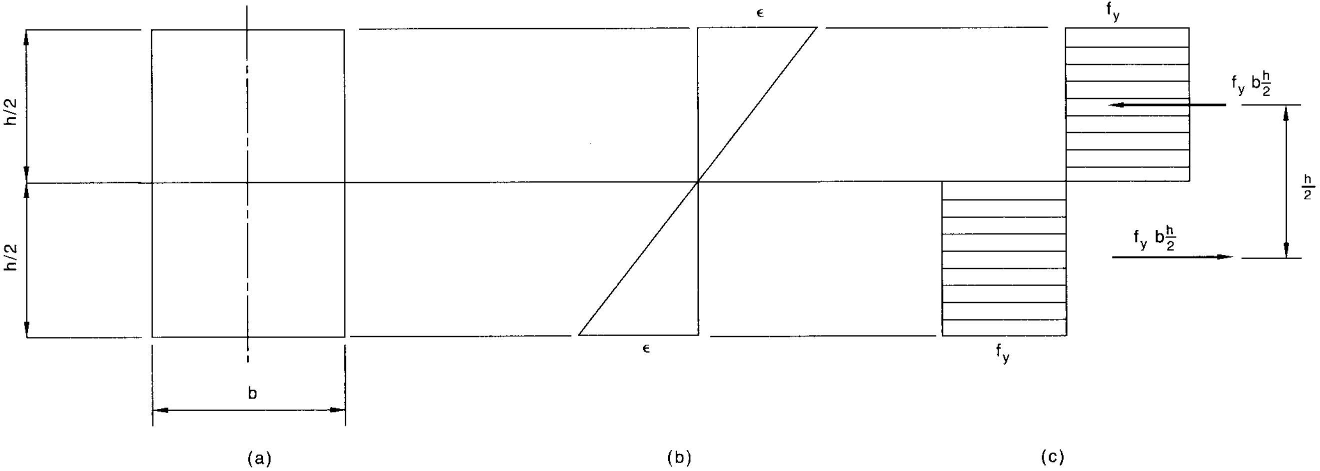


FIGURE 3.24 For a rectangular beam (*a*) in the plastic range, strain distribution (*b*) is linear, while stress distribution (*c*) is rectangular.

To determine these stresses, consider the portion of a beam with length dx between vertical sections 1–1 and 2–2 (Fig. 3.25). At a horizontal section a distance y from the neutral axis, the horizontal shear force $\Delta H(y)$ equals the difference between the normal forces acting above the section on the two faces:

$$\Delta H(y) = \int_y^{c_t} f_2(y)b(y)\,dy - \int_y^{c_t} f_1(y)b(y)\,dy \tag{3.67}$$

where $f_2(y)$ and $f_1(y)$ are the bending-stress distributions at sections 2–2 and 1–1, respectively.

If the bending stresses vary linearly with depth, then, according to Eq. (3.61),

$$f_2(y) = \frac{M_2 y}{I} \tag{3.68a}$$

$$f_1(y) = \frac{M_1 y}{I} \tag{3.68b}$$

where M_2 and M_1 are the internal bending moments at sections 2–2 and 1–1, respectively, and I is the moment of inertia about the neutral axis of the beam cross section. Substitution in Eq. (3.67) gives

$$\Delta H(y) = \frac{M_2 - M_1}{I} \int_y^{c_t} yb(y)\,dy = \frac{Q(y)}{I}\,dM \tag{3.69}$$

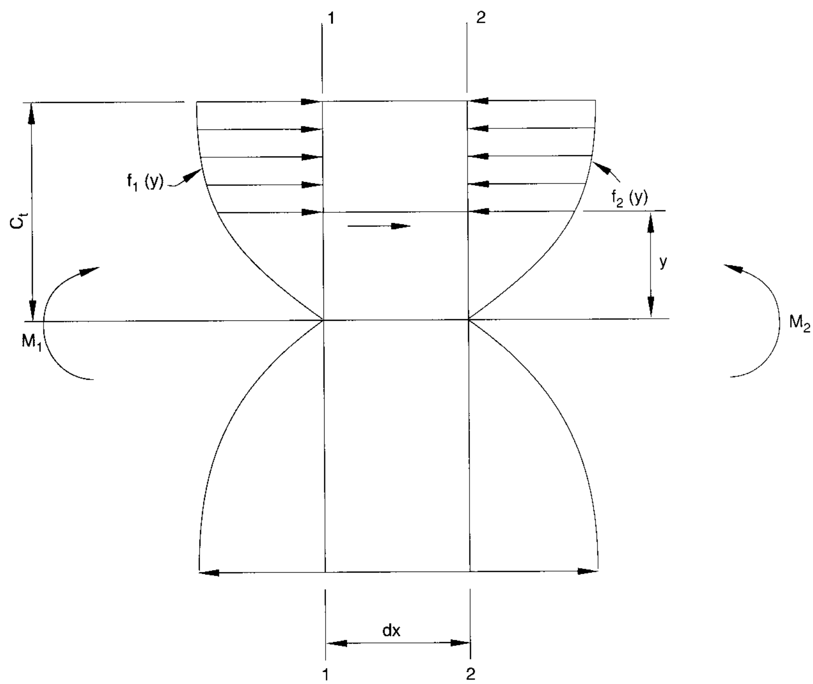


FIGURE 3.25 Shear stresses in a beam.

where $Q(y) = \int_y^{c_t} yb(y)\,dy$ = **static moment** about neutral axis of the area above the plane at a distance y from the neutral axis
$b(y)$ = width of beam
$dM = M_2 - M_1$

Division of $\Delta H(y)$ by the area $b(y)$ dx yields the shear stress at y:

$$v(y) = \frac{\Delta H(y)}{b(y)\,dx} = \frac{Q(y)}{Ib(y)}\frac{dM}{dx} \tag{3.70}$$

Integration of $v(y)$ over the cross section provides the total internal vertical shear force V on the section:

$$V = \int_{c_b}^{c_t} v(y)b(y)\,dy \tag{3.71}$$

To satisfy equilibrium requirements, V must be equal in magnitude but opposite in direction to the shear at the section due to the loading.

Substitution of Eq. 3.70 in Eq. 3.71 gives

$$V = \int_{c_b}^{c_t} \frac{Q(y)}{Ib(y)}\frac{dM}{dx}\,b(y)\,dy = \frac{dM}{dx}\frac{1}{I}\int_{c_b}^{c_t} Q(y)\,dy = \frac{dM}{dx} \tag{3.72}$$

inasmuch as $I = \int_{c_b}^{c_t} Q(y)\,dy$. Equation (3.72) indicates that shear is the rate of change of bending moment along the span of the beam.

Substitution of Eq. (3.72) into Eq. (3.70) yields an expression for calculating the shear stress at any section depth:

$$v(y) = \frac{VQ(y)}{Ib(y)} \tag{3.73}$$

According to Eq. (3.73), the maximum shear stress occurs at a depth y when the ratio $Q(y)/b(y)$ is maximum.

For rectangular cross sections, the maximum shear stress occurs at middepth and equals

$$v_{\max} = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A} \tag{3.74}$$

where h is the beam depth and A is the cross-sectional area.

3.18 SHEAR, MOMENT, AND DEFORMATION RELATIONSHIPS IN BEAMS

The relationship between shear and moment identified in Eq. (3.72), that is, $V = dM/dx$, indicates that the shear force at a section is the rate of change of the bending moment. A similar relationship exists between the load on a beam and the shear at a section. Figure 3.26*b* shows the resulting internal forces and moments for the portion of beam dx shown in Fig. 3.26*a*. Note that when the internal shear acts upward on the left of the section, the shear is positive; and when the shear acts upward on the right of the section, it is negative. For equilibrium of the vertical forces,

$$\Sigma F_y = V - (V + dV) + w(x)\,dx = 0 \tag{3.75}$$

Solving for $w(x)$ gives

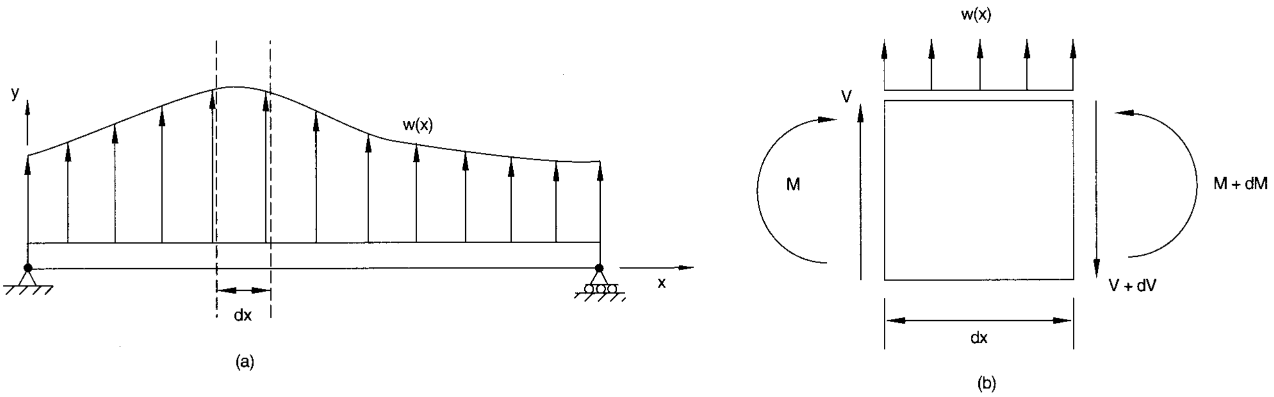


FIGURE 3.26 (*a*) Beam with distributed loading. (*b*) Internal forces and moments on a section of the beam.

$$w(x) = \frac{dV}{dx} \tag{3.76}$$

This equation indicates that the rate of change in shear at any section equals the load per unit length at that section. When concentrated loads act on a beam, Eqs. (3.72) and (3.76) apply to the region of the beam between the concentrated loads.

Beam Deflections. To this point, only relationships between the load on a beam and the resulting internal forces and stresses have been established. To calculate the deflection at various points along a beam, it is necessary to know the relationship between load and the deformed curvature of the beam or between bending moment and this curvature.

When a beam is subjected to loads, it deflects. The deflected shape of the beam taken at the neutral axis may be represented by an elastic curve $\delta(x)$. If the slope of the deflected shape is such that $d\delta/dx << 1$, the radius of curvature R at a point x along the span is related to the derivatives of the ordinates of the elastic curve $\delta(x)$ by

$$\frac{1}{R} = \frac{d^2\delta}{dx^2} = \frac{d}{dx}\left(\frac{d\delta}{dx}\right) = \phi \tag{3.77}$$

$1/R$ is referred to as the **curvature** ϕ of a beam. It represents the rate of change of the slope $\phi = d\delta/dx$ of the neutral axis.

Consider the deformation of the dx portion of a beam shown in Fig. 3.26*b*. Before the loads act, sections 1–1 and 2–2 are vertical (Fig. 3.27*a*). After the loads act, assuming plane sections remain plane, this portion becomes trapezoidal. The top of the beam shortens an amount $\varepsilon_t\, dx$ and the beam bottom an amount $\varepsilon_b\, dx$, where ε_t is the compressive unit strain at the beam top and ε_b is the tensile unit strain at the beam bottom. Each side rotates through a small angle. Let the angle of rotation of section 1–1 be $d\theta_1$ and that of section 2–2, $d\theta_2$ (Fig. 3.27*b*). Hence the angle between the two faces will be $d\theta_1 + d\theta_2 = d\theta$. Since $d\theta_1$ and $d\theta_2$ are small, the total shortening of the beam top between sections 1–1 and 2–2 is also given by $c_t\, d\theta = \varepsilon_t\, dx$, from which $d\theta/dx = \varepsilon_t/c_t$, where c_t is the distance from the neutral axis to the beam top. Similarly, the total lengthening of the beam bottom is given by $c_b\, d\theta = \varepsilon_b\, dx$, from which $d\theta/dx = \varepsilon_b/c_b$, where c_b is the distance from the neutral axis to the beam bottom. By definition, the beam curvature is therefore given by

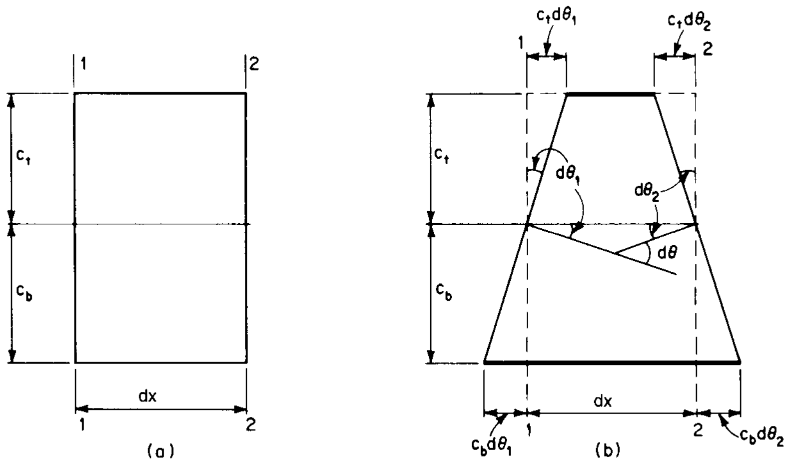


FIGURE 3.27 (*a*) Portion of an unloaded beam. (*b*) Deformed portion after beam is loaded.

$$\phi = \frac{d}{dx}\left(\frac{d\delta}{dx}\right) = \frac{d\theta}{dx} = \frac{\varepsilon_t}{c_t} = \frac{\varepsilon_b}{c_b} \tag{3.78}$$

When the stress-strain diagram for the material is linear, $\varepsilon_t = f_t/E$ and $\varepsilon_b = f_b/E$, where f_t and f_b are the unit stresses at top and bottom surfaces and E is the modulus of elasticity. By Eq. (3.60), $f_t = M(x)c_t/I(x)$ and $f_b = M(x)c_b/I(x)$, where x is the distance along the beam span where the section dx is located and $M(x)$ is the moment at the section. Substitution for ε_t and f_t or ε_b and f_b in Eq. (3.78) gives

$$\phi = \frac{d^2\delta}{dx^2} = \frac{d}{dx}\left(\frac{d\delta}{dx}\right) = \frac{d\theta}{dx} = \frac{M(x)}{EI(x)} \tag{3.79}$$

Equation (3.79) is of fundamental importance, for it relates the internal bending moment along the beam to the curvature or second derivative of the elastic curve $\delta(x)$, which represents the deflected shape. Equations (3.72) and (3.76) further relate the bending moment $M(x)$ and shear $V(x)$ to an applied distributed load $w(x)$. From these three equations, the following relationships between load on the beam, the resulting internal forces and moments, and the corresponding deformations can be shown:

$$\delta(x) = \text{elastic curve representing the deflected shape} \tag{3.80a}$$

$$\frac{d\delta}{dx} = \theta(x) = \text{slope of the deflected shape} \tag{3.80b}$$

$$\frac{d^2\delta}{dx^2} = \phi = \frac{M(x)}{EI(x)} = \text{curvature of the deflected shape and also the moment-curvature relationship} \tag{3.80c}$$

$$\frac{d^3\delta}{dx^3} = \frac{d}{dx}\left[\frac{M(x)}{EI(x)}\right] = \frac{V(x)}{EI(x)} = \text{shear-deflection relationship} \tag{3.80d}$$

$$\frac{d^4\delta}{dx^4} = \frac{d}{dx}\left[\frac{V(x)}{EI(x)}\right] = \frac{w(x)}{EI(x)} = \text{load-deflection relationship} \tag{3.80e}$$

These relationships suggest that the shear force, bending moment, and beam slope and deflection may be obtained by integrating the load distribution. For some simple cases this approach can be used conveniently. However, it may be cumbersome when a large number of concentrated loads act on a structure. Other methods are suggested in Arts. 3.32 to 3.39.

Shear, Moment, and Deflection Diagrams. Figures 3.28 to 3.49 show some special cases in which shear, moment, and deformation distributions can be expressed in analytic form. The figures also include diagrams indicating the variation of shear, moment, and deformations along the span. A diagram in which shear is plotted along the span is called a **shear diagram.** Similarly, a diagram in which bending moment is plotted along the span is called a **bending-moment diagram.**

Consider the simply supported beam subjected to a downward-acting, uniformly distributed load w (units of load per unit length) in Fig. 3.31*a*. The support reactions R_1 and R_2 may be determined from equilibrium equations. Summing moments about the left end yields

$$\Sigma M = R_2L - wL\frac{L}{2} = 0 \qquad R_2 = \frac{wL}{2}$$

R_1 may then be found from equilibrium of vertical forces:

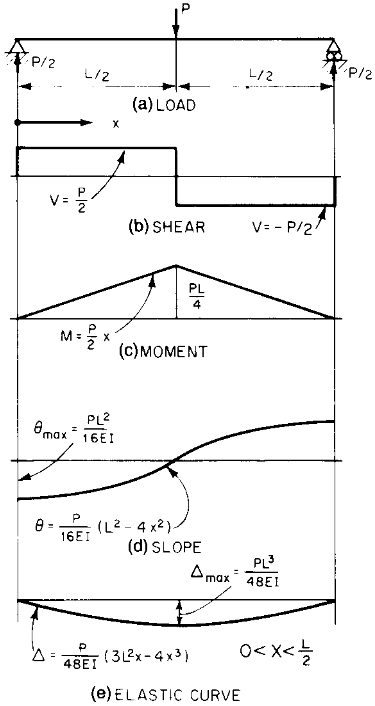


FIGURE 3.28 Shears moments, and deformations for midspan load on a simple beam.

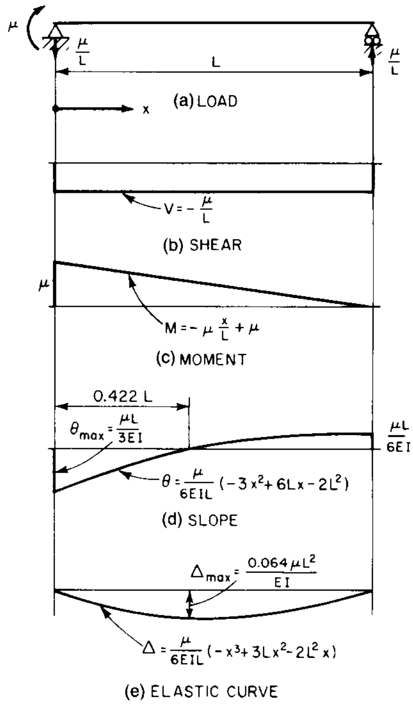


FIGURE 3.29 Diagrams for moment applied at one end of a simple beam.

$$\Sigma F_y = R_1 + R_2 - wL = 0 \qquad R_1 = \frac{wL}{2}$$

With the origin taken at the left end of the span, the shear at any point can be obtained from Eq. (3.80*e*) by integration: $V = \int -w\,dx = -wx + C_1$, where C_1 is a constant. When $x = 0$, $V = R_1 = wL/2$, and when $x = L$, $V = -R_2 = -wL/2$. For these conditions to be satisfied, $C_1 = wL/2$. Hence the equation for shear is $V(x) = -wx + wL/2$ (Fig. 3.31*b*).

The bending moment at any point is, by Eq. (3.80*d*), $M(x) = \int V\,dx = \int(-wx + wL/2)\,dx = -wx^2/2 + wLx/2 + C_2$, where C_2 is a constant. In this case, when $x = 0$, $M = 0$. Hence $C_2 = 0$, and the equation for bending moment is $M(x) = \frac{1}{2}w(-x^2 + Lx)$, as shown in Fig. 3.31*c*. The maximum bending moment occurs at midspan, where $x = L/2$, and equals $wL^2/8$.

From Eq. (3.80*c*), the slope of the deflected member at any point along the span is

$$\theta(x) = \int \frac{M(x)}{EI}\,dx = \int \frac{w}{2EI}(-x^2 + Lx)\,dx = \frac{w}{2EI}\left(-\frac{x^3}{3} + \frac{Lx^2}{2}\right) + C_3$$

where C_3 is a constant. When $x = L/2$, $\theta = 0$. Hence $C_3 = -wL^3/24EI$, and the equation for slope is

$$\theta(x) = \frac{w}{24EI}(-4x^3 + 6Lx^2 - L^3)$$

(See Fig. 3.31*d*.)

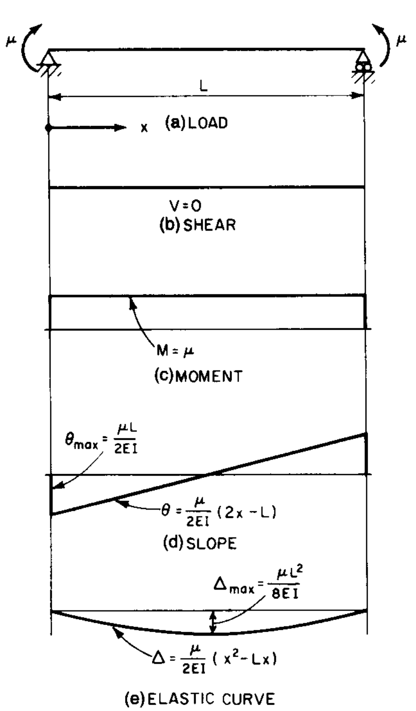


FIGURE 3.30 Diagrams for moments applied at both ends of a simple beam.

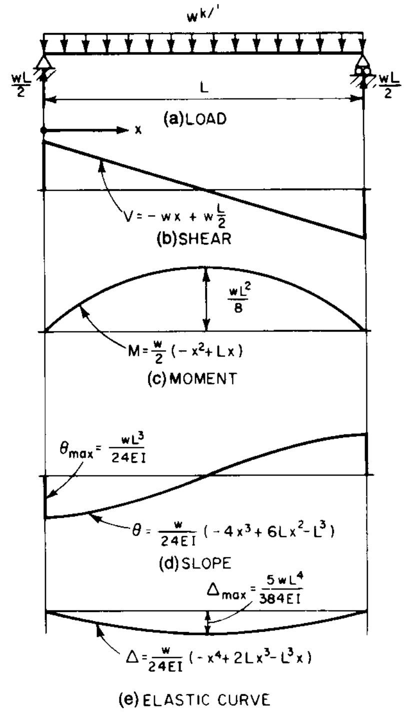


FIGURE 3.31 Shears, moments, and deformations for uniformly loaded simple beam.

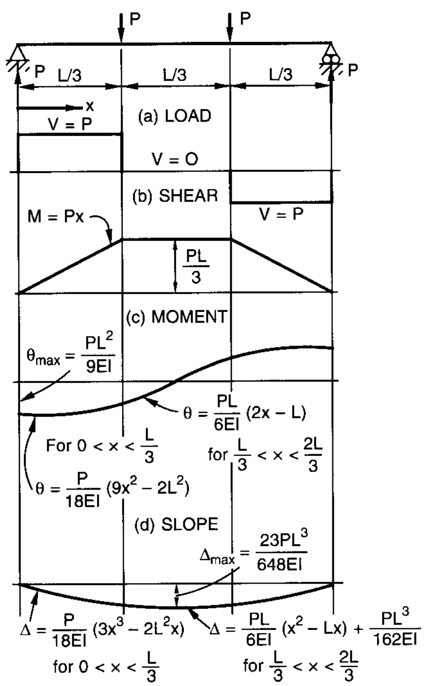


FIGURE 3.32 Simple beam with concentrated load at the third points.

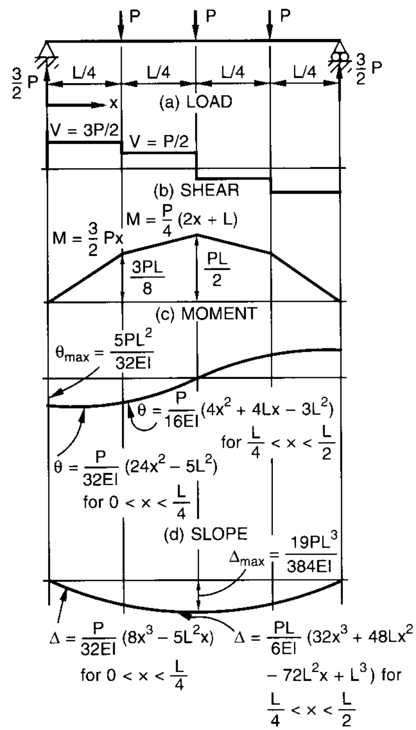


FIGURE 3.33 Diagrams for simple beam loaded at quarter points.

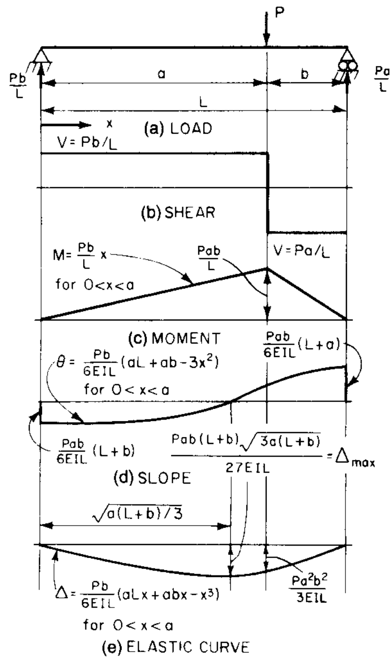


FIGURE 3.34 Diagrams for concentrated load on a simple beam.

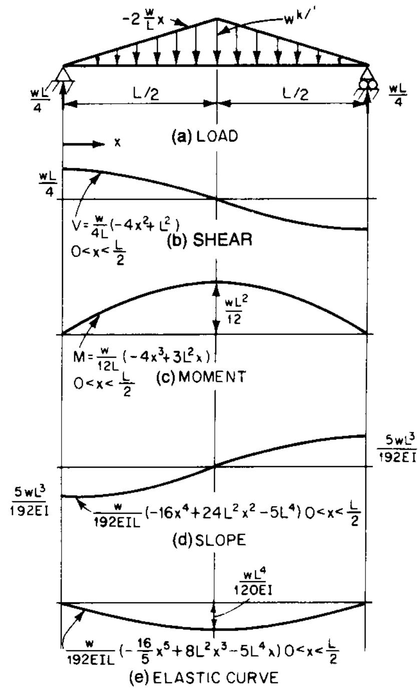


FIGURE 3.35 Symmetrical triangular load on a simple beam.

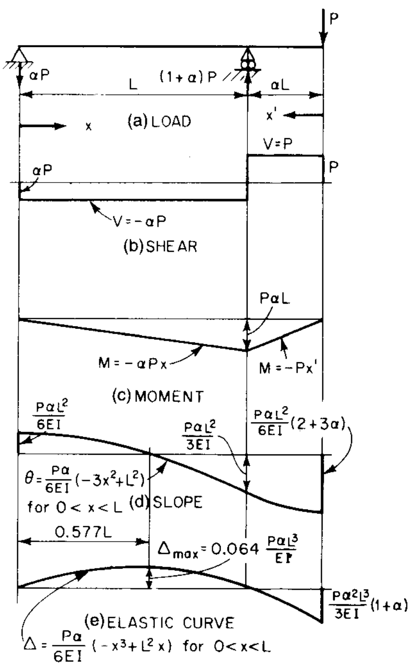


FIGURE 3.36 Concentrated load on a beam overhang.

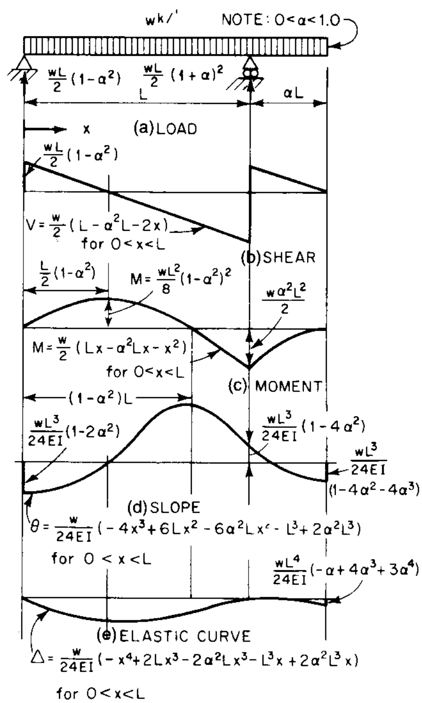


FIGURE 3.37 Uniformly loaded beam with overhang.

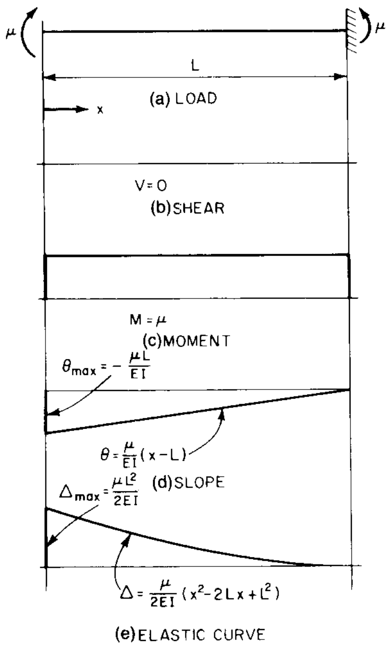


FIGURE 3.38 Shears, moments, and deformations for moment at one end of a cantilever.

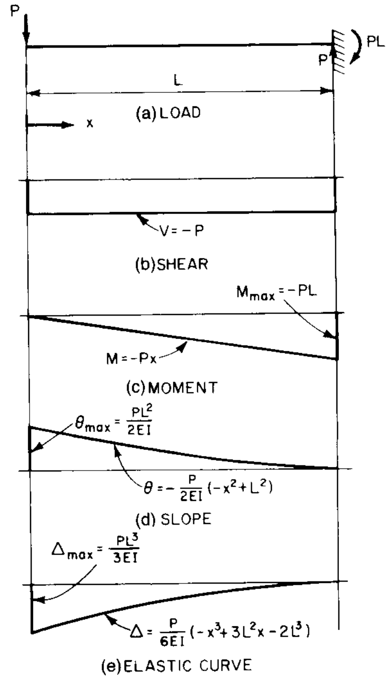


FIGURE 3.39 Diagrams for concentrated load on a cantilever.

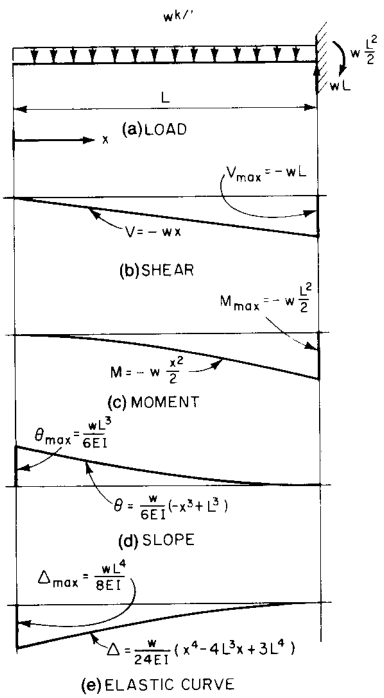


FIGURE 3.40 Shears, moments, and deformations for uniformly loaded cantilever.

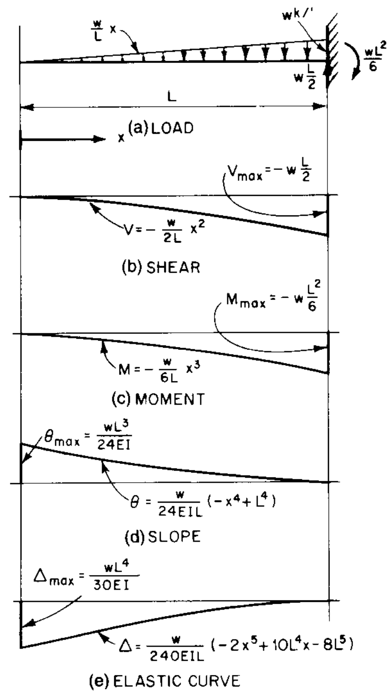


FIGURE 3.41 Triangular load on cantilever with maximum at support

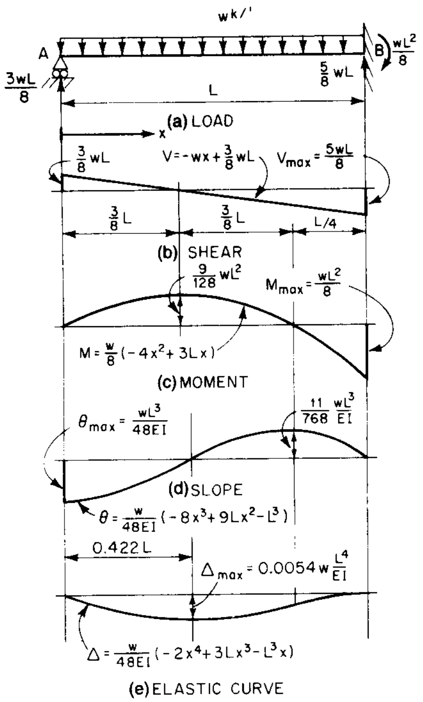


FIGURE 3.42 Uniform load on beam with one end fixed, one end on rollers.

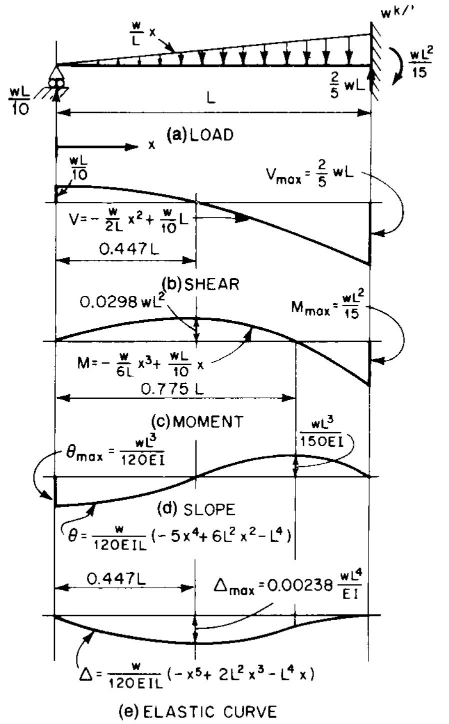


FIGURE 3.43 Triangular load on beam with one end fixed, one end on rollers.

The deflected-shape curve at any point is, by Eq. (3.80*b*),

$$\delta(x) = \frac{w}{24EI}\int(-4x^3 + 6Lx^2 - L^3)\,dx$$

$$= -wx^4/24EI + wLx^3/12EI - wL^3x/24EI + C_4$$

where C_4 is a constant. In this case, when $x = 0$, $\delta = 0$. Hence $C_4 = 0$, and the equation for deflected shape is

$$\delta(x) = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x)$$

as shown in Fig. 3.31*e*. The maximum deflection occurs at midspan, where $x = L/2$, and equals $-5wL^4/384EI$.

For concentrated loads, the equations for shear and bending moment are derived in the region between the concentrated loads, where continuity of these diagrams exists. Consider the simply supported beam subjected to a concentrated load at midspan (Fig. 3.28*a*). From equilibrium equations, the reactions R_1 and R_2 equal $P/2$. With the origin taken at the left end of the span, $w(x) = 0$ when $x \neq L/2$. Integration of Eq. (3.80*e*) gives $V(x) = C_3$, a constant, for $x = 0$ to $L/2$, and $V(x) = C_4$, another constant, for $x = L/2$ to L. Since $V = R_1 = P/2$ at $x = 0$, $C_3 = P/2$. Since $V = -R_2 = -P/2$ at $x = L$, $C_4 = -P/2$. Thus, for $0 \le x < L/2$, $V = P/2$, and for $L/2 < x \le L$, $V = -P/2$ (Fig. 3.28*b*). Similarly, equations

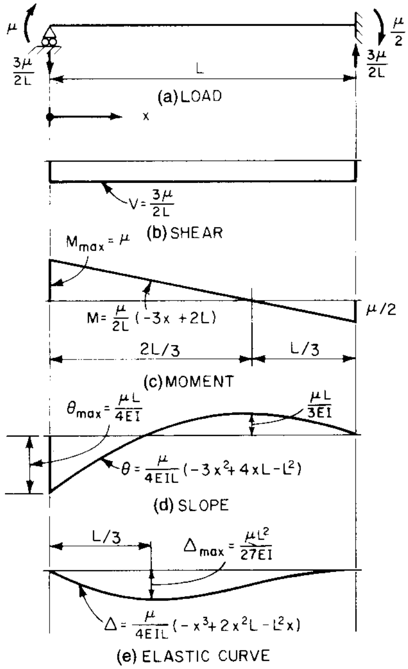


FIGURE 3.44 Moment applied at one end of a beam with a fixed end.

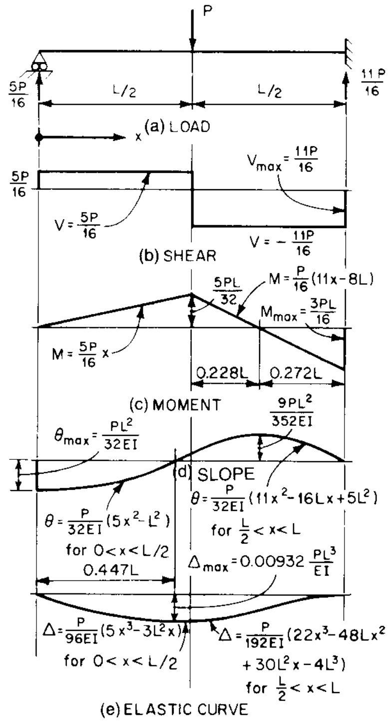


FIGURE 3.45 Load at midspan of beam with one fixed end, one end on rollers.

for bending moment, slope, and deflection can be expressed from $x = 0$ to $L/2$ and again for $x = L/2$ to L, as shown in Figs. 3.28*c*, 3.28*d*, and 3.28*e*, respectively.

In practice, it is usually not convenient to derive equations for shear and bending-moment diagrams for a particular loading. It is generally more convenient to use equations of equilibrium to plot the shears, moments, and deflections at critical points along the span. For example, the internal forces at the quarter span of the uniformly loaded beam in Fig. 3.31 may be determined from the free-body diagram in Fig. 3.50. From equilibrium conditions for moments about the right end,

$$\sum M = M + \left(\frac{wL}{4}\right)\left(\frac{L}{8}\right) - \left(\frac{wL}{2}\right)\left(\frac{L}{4}\right) = 0 \tag{3.81a}$$

$$M = \frac{3wL^2}{32} \tag{3.81b}$$

Also, the sum of the vertical forces must equal zero:

$$\sum F_y = \frac{wL}{2} - \frac{wL}{4} - V = 0 \tag{3.82a}$$

$$V = \frac{wL}{4} \tag{3.82b}$$

Several important concepts are demonstrated in the preceding examples:

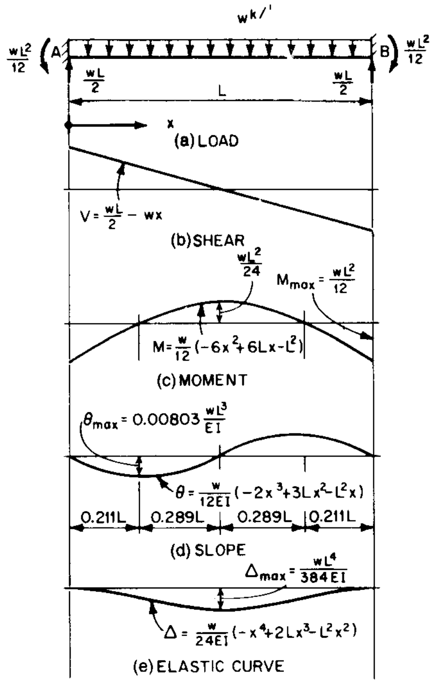


FIGURE 3.46 Shears, moments, and deformations for uniformly loaded fixed-end beam.

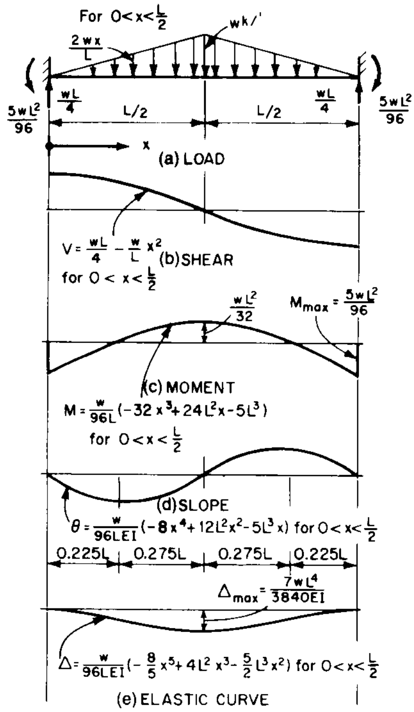


FIGURE 3.47 Diagrams for triangular load on a fixed-end beam.

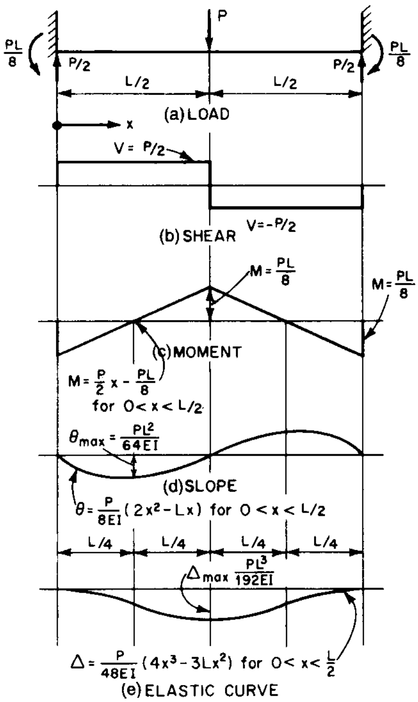


FIGURE 3.48 Shears, moments, and deformations for load at midspan of a fixed-end beam.

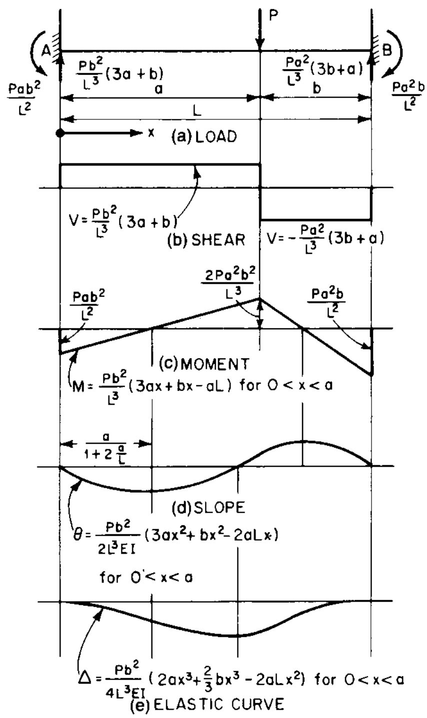


FIGURE 3.49 Diagrams for concentrated load on a fixed-end beam.

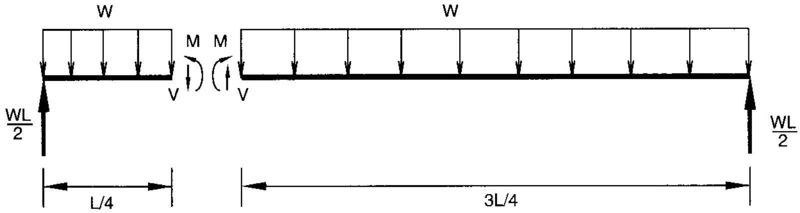


FIGURE 3.50 Bending moment and shear at quarter point of a uniformly loaded simple beam.

- The shear at a section is the algebraic sum of all forces on either side of the section.
- The bending moment at a section is the algebraic sum of the moments about the section of all forces and applied moments on either side of the section.
- A maximum bending moment occurs where the shear or slope of the bending-moment diagram is zero.
- Bending moment is zero where the slope of the elastic curve is at maximum or minimum.
- Where there is no distributed load along a span, the shear diagram is a horizontal line. (Shear is a constant, which may be zero.)
- The shear diagram changes sharply at the point of application of a concentrated load.
- The differences between the bending moments at two sections of a beam equals the area under the shear diagram between the two sections.
- The difference between the shears at two sections of a beam equals the area under the distributed load diagram between those sections.

3.19 SHEAR DEFLECTIONS IN BEAMS

Shear deformations in a beam add to the deflections due to bending discussed in Art. 3.18. Deflections due to shear are generally small, but in some cases they should be taken into account.

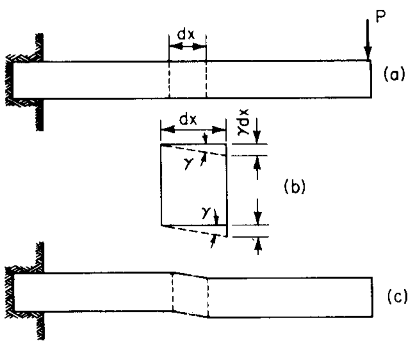


FIGURE 3.51 (*a*) Cantilever with a concentrated load. (*b*) Shear deformation of a small portion of the beam. (*c*) Shear deflection of the cantilever.

When a cantilever is subjected to load P (Fig. 3.51*a*), a portion dx of the span undergoes a shear deformation (Fig. 3.51*b*). For an elastic material, the angle γ equals the ratio of the shear stress v to the shear modulus of elasticity G. Assuming that the shear on the element is distributed uniformly, which is an approximation, the deflection of the beam $d\delta_s$ caused by the deformation of the element is

$$d\delta_s = \gamma\, dx = \frac{v}{G}\, dx \approx \frac{V}{AG}\, dx \qquad (3.83)$$

Figure 3.52*c* shows the corresponding shear deformation. The total shear deformation at the free end of a cantilever is

$$\delta_s \approx \int_0^L \frac{V}{AG}\, dx = \frac{PL}{AG} \tag{3.84}$$

The shear deflection given by Eq. (3.84) is usually small compared with the flexural deflection for different materials and cross-sectional shapes. For example, the flexural deflection at the free end of a cantilever is $\delta_f = PL^3/3EI$. For a rectangular section made of steel with $G \approx 0.4E$, the ratio of shear deflection to flexural deflection is

$$\frac{\delta_s}{\delta_f} = \frac{PL/AG}{PL^3/3EI} = \frac{5}{8}\left(\frac{h}{L}\right)^2 \tag{3.85}$$

where h = depth of the beam. Thus, for a beam of rectangular section when $h/L = 0.1$, the shear deflection is less than 1% of the flexural deflection.

Shear deflections can be approximated for other types of beams in a similar way. For example, the midspan shear deflection for a simply supported beam loaded with a concentrated load at the center is $PL/4AG$.

3.20 MEMBERS SUBJECTED TO COMBINED FORCES

Most of the relationships presented in Arts. 3.16 to 3.19 hold only for symmetrical cross sections, e.g., rectangles, circles, and wide-flange beams, and only when the plane of the loads lies in one of the axes of symmetry. There are several instances where this is not the case, e.g., members subjected to axial load and bending and members subjected to torsional loads and bending.

Combined Axial Load and Bending. For short, stocky members subjected to both axial load and bending, stresses may be obtained by superposition if (1) the deflection due to bending is small and (2) all stresses remain in the elastic range. For these cases, the total stress normal to the section at a point equals the algebraic sum of the stress due to axial load and the stress due to bending about each axis:

$$f = \pm\frac{P}{A} \pm \frac{M_x}{S_x} \pm \frac{M_y}{S_y} \tag{3.86}$$

where P = axial load
A = cross-sectional area
M_x = bending moment about the centroidal x axis
S_x = elastic section modulus about the centoidal x axis
M_y = bending moment about the centroidal y axis
S_y = elastic section modulus about the centroidal y axis

If bending is only about one axis, the maximum stress occurs at the point of maximum moment. The two signs for the axial and bending stresses in Eq. (3.86) indicate that when the stresses due to the axial load and bending are all in tension or all in compression, the terms should be added. Otherwise, the signs should be obeyed when performing the arithmetic. For convenience, compressive stresses can be taken as negative and tensile stresses as positive.

Bending and axial stress are often caused by eccentrically applied axial loads. Figure 3.52 shows a column carrying a load P with eccentricity e_x and e_y. The stress in this case may be found by incorporating the resulting moments $M_x = Pe_x$ and $M_y = Pe_y$ into Eq. (3.86).

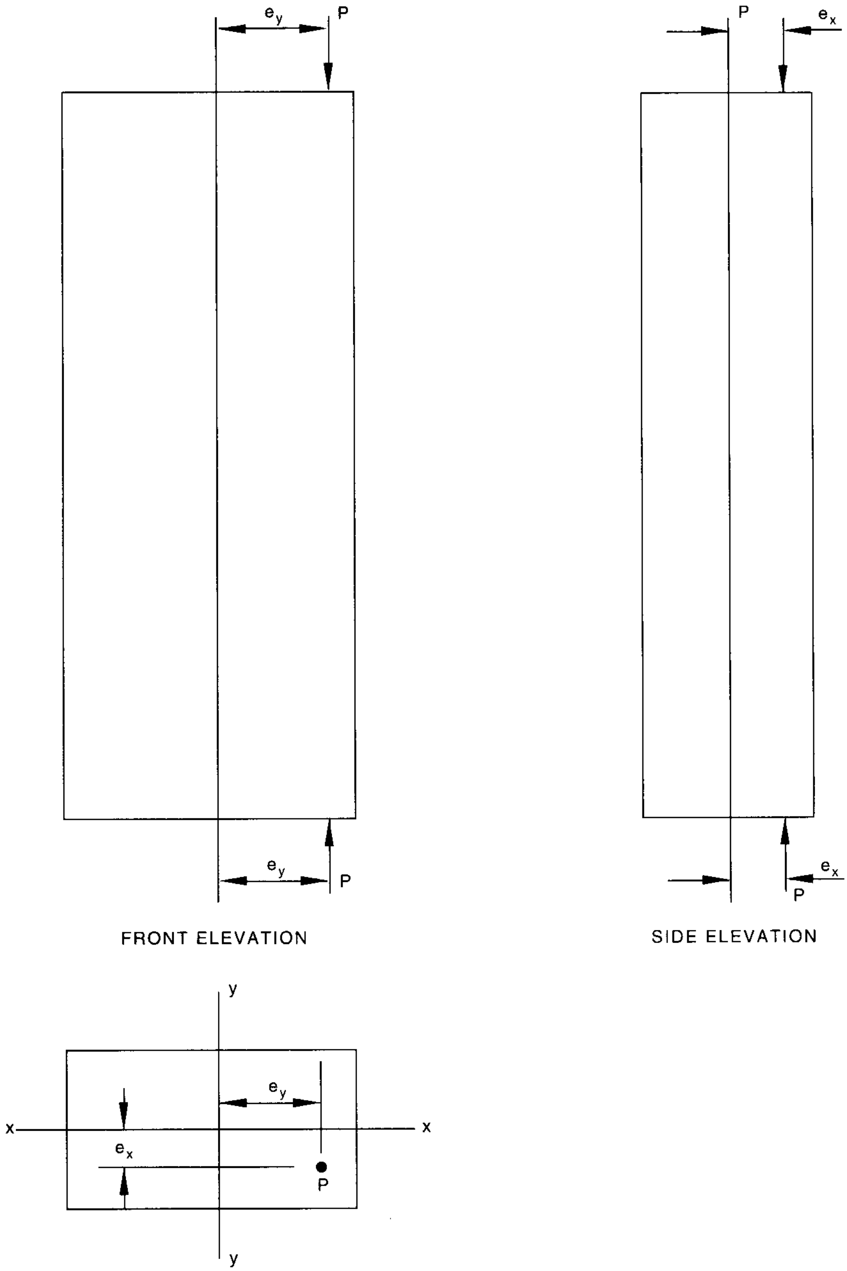


FIGURE 3.52 Eccentrically loaded column.

If the deflection due to bending is large, M_x and M_y should include the additional moment produced by second-order effects. Methods for incorporating these effects are presented in Arts. 3.46 to 3.48.

3.21 UNSYMMETRICAL BENDING

When the plane of loads acting transversely on a beam does not contain any of the beam's axes of symmetry, the loads may tend to produce twisting as well as bending. Figure 3.53 shows a horizontal channel twisting even though the vertical load H acts through the centroid of the section.

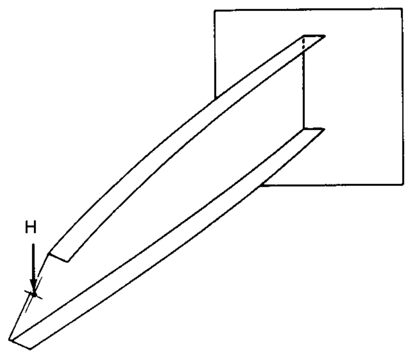


FIGURE 3.53 Twisting of a channel.

The **bending axis** of a beam is the longitudinal line through which transverse loads should pass to preclude twisting as the beam bends. The **shear center** for any section of the beam is the point in the section through which the bending axis passes.

For sections having two axes of symmetry, the shear center is also the centroid of the section. If a section has an axis of symmetry, the shear center is located on that axis but may not be at the centroid of the section. Figure 3.54 shows a channel section in which the horizontal axis is the axis of sym-

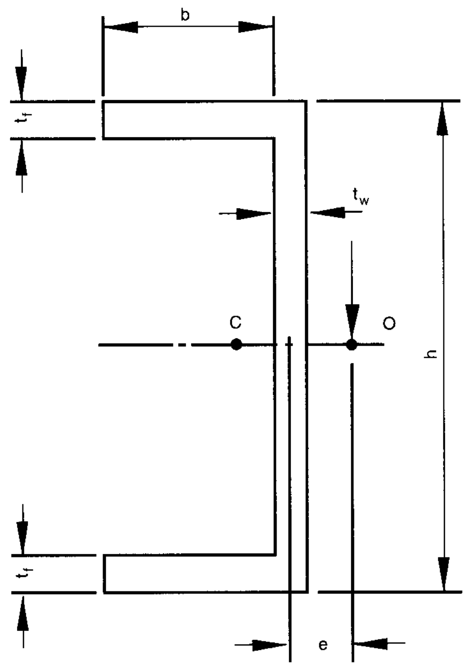


FIGURE 3.54 Relative position of shear center O and centroid C of a channel.

metry. Point O represents the shear center. It lies on the horizontal axis but not at the centroid C. A load at the section must pass through the shear center if twisting of the member is not to occur. The location of the shear center relative to the center of the web can be obtained from

$$e = \frac{b/2}{1 + \frac{1}{6}(A_w/A_f)} \tag{3.87}$$

where b = width of flange overhang
$A_f = t_f b$ = area of flange overhang
$A_w = t_w h$ = web area

(F. Bleich, *Buckling Strength of Metal Structures,* McGraw-Hill, Inc., New York.)

For a member with an unsymmetrical cross section subject to combined axial load and biaxial bending, Eq. (3.86) must be modified to include the effects of unsymmetrical bending. In this case, stress in the elastic range is given by

$$f = \frac{P}{A} + \frac{M_y - M_x(I_{xy}/I_x)}{I_y - (I_{xy}/I_x)I_{xy}}\, x + \frac{M_x - M_y(I_{xy}/I_y)}{I_x - (I_{xy}/I_y)I_{xy}}\, y \tag{3.88}$$

where A = cross-sectional area
M_x, M_y = bending moment about x–x and y–y axes
I_x, I_y = moment of inertia about x–x and y–y axes
x, y = distance of stress point under consideration from y–y and x–x axes
I_{xy} = product of inertia

$$I_{xy} = \int xy\, dA \tag{3.89}$$

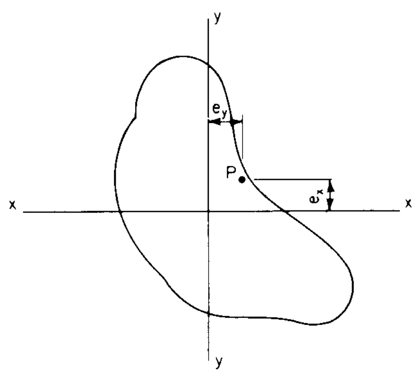


FIGURE 3.55 Eccentric load P on an unsymmetrical cross section.

Moments M_x and M_y may be caused by transverse loads or eccentricities of axial loads. An example of the latter case is shown in Fig. 3.55. For an axial load P, $M_x = Pe_x$ and $M_y = Pe_y$, where e_x and e_y are eccentricities with respect to the x–x and y–y axes, respectively.

To show an application of Eq. (3.88) to an unsymmetrical section, stresses in the lintel angle in Fig. 3.56 will be calculated for $M_x = 200$ in-kips, $M_y = 0$, and $P = 0$. The centroidal axes x–x and y–y are 2.6 and 1.1 in from the bottom and left side, respectively, as shown in Fig. 3.56. The moments of inertia are $I_x = 47.82$ in^4 and $I_y = 11.23$ in^4. The product of inertia can be calculated by dividing the angle into two rectangular parts and then applying Eq. (3.89):

$$\begin{aligned} I_{xy} &= \int xy\, dA = A_1x_1y_1 + A_2x_2y_2 \\ &= 7(-0.6)(0.9) + 3(1.4)(-2.1) = -12.6 \end{aligned} \tag{3.90}$$

where A_1 and A_2 = cross-sectional areas of parts 1 and 2
x_1 and x_2 = horizontal distance from the angle's centroid to the centroid of parts 1 and 2

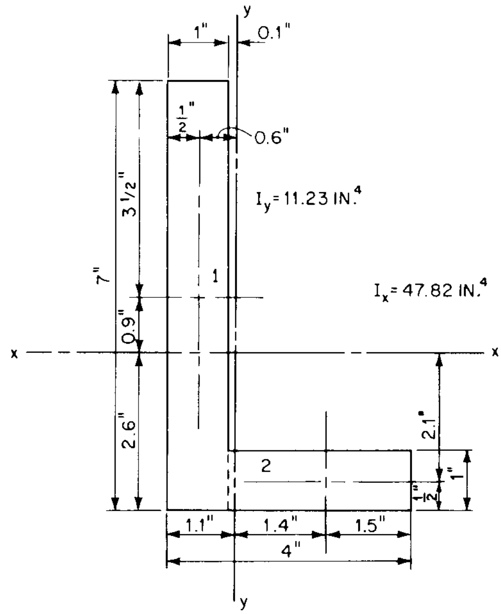


FIGURE 3.56 Steel lintel angle.

y_1 and y_2 = vertical distance from the angle's centroid to the centroid of parts 1 and 2

Substitution in Eq. (3.88) gives

$$f = 6.64x + 5.93y$$

This equation indicates that the maximum stresses normal to the cross section occur at the corners of the angle. A maximum compressive stress of 25.43 ksi occurs at the upper right corner, where $x = -0.1$ and $y = 4.4$. A maximum tensile stress of 22.72 ksi occurs at the lower left corner, where $x = -1.1$ and $y = -2.6$.

(I. H. Shames, *Mechanics of Deformable Solids,* Prentice-Hall, Inc., Englewood Cliffs, N.J.; F.R. Shanley, *Strength of Materials,* McGraw-Hill, Inc., New York.)

CONCEPTS OF WORK AND ENERGY

The concepts of work and energy are often useful in structural analysis. These concepts provide a basis for some of the most important theorems of structural analysis.

3.22 WORK OF EXTERNAL FORCES

Whenever a force is displaced by a certain amount or a displacement is induced by a certain force, **work** is generated. The increment of work done on a body by a force **F** during an incremental displacement **ds** from its point of application is

$$dW = \mathbf{F}\,\mathbf{ds}\cos\alpha \tag{3.91}$$

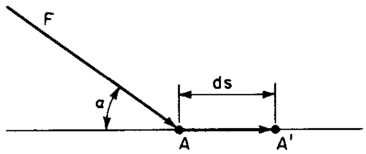


FIGURE 3.57 Force performs work in direction of displacement.

where α is the angle between **F** and **ds** (Fig. 3.57). Equation (3.91) implies that work is the product of force and the component of displacement in the line of action of the force, or the product of displacement and the component of force along the path of the displacement. If the component of the displacement is in the same direction as the force or the component of the force acts in the same direction as the path of displacement, the work is positive; otherwise, the work is negative. When the line of action of the force is perpendicular to the direction of displacement ($\alpha = \pi/2$), no work is done.

When the displacement is a finite quantity, the total work can be expressed as

$$W = \int F\cos\alpha\, ds \tag{3.92}$$

Integration is carried out over the path the force travels, which may not be a straight line.

The work done by the weight of a body, which is the force, when it is moved in a vertical direction is the product of the weight and vertical displacement. According to Eq. (3.91) and with α the angle between the downward direction of gravity and the imposed displacement, the weight does positive work when movement is down. It does negative work when movement is up.

In a similar fashion, the rotation of a body by a moment **M** through an incremental angle $\mathbf{d\theta}$ also generates work. The increment of work done in this case is

$$dW = M\,d\theta \tag{3.93}$$

The total work done during a finite angular displacement is

$$W = \int M\,d\theta \tag{3.94}$$

3.23 VIRTUAL WORK AND STRAIN ENERGY

Consider a body of negligible dimensions subjected to a force **F.** Any displacement of the body from its original position will create work. Suppose a small displacement $\boldsymbol{\delta}\mathbf{s}$ is assumed but does not actually take place. This displacement is called a **virtual displacement,** and the work $\boldsymbol{\delta}W$ done by force **F** during the displacement $\boldsymbol{\delta}\mathbf{s}$ is called **virtual work.** Virtual work also is done when a virtual force $\boldsymbol{\delta}\mathbf{F}$ acts over a displacement **s.**

Virtual Work on a Particle. Consider a particle at location A that is in equilibrium under the concurrent forces $\mathbf{F_1}$, $\mathbf{F_2}$, and $\mathbf{F_3}$ (Fig. 3.58*a*). Hence equilibrium requires that the sum of the components of the forces along the x axis by zero:

$$\Sigma F_x = F_1\cos\alpha_1 - F_2\cos\alpha_2 + F_3\cos\alpha_3 = 0 \tag{3.95}$$

where α_1, α_2, α_3 = angle force makes with the x axis. If the particle is displaced a virtual amount $\boldsymbol{\delta}\mathbf{s}$ along the x axis from A to A' (Fig. 3.58*b*), then the total virtual work done by the forces is the sum of the virtual work generated by displacing each of the forces $\mathbf{F_1}$, $\mathbf{F_2}$, and $\mathbf{F_3}$. According to Eq. (3.91),

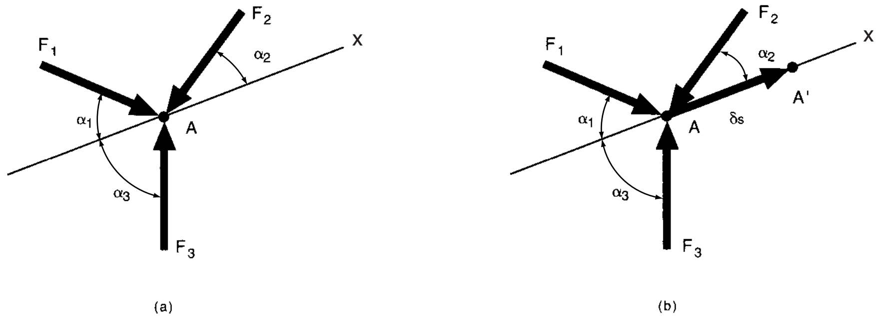


FIGURE 3.58 (*a*) Forces act on a particle *A*. (*b*) Forces perform virtual work over virtual displacement δ_s.

$$\delta W = F_1 \cos \alpha_2 \; \delta s - F_2 \cos \alpha_2 \; \delta s + F_3 \cos \alpha_3 \; \delta s \qquad (3.96)$$

Factoring δs from the right side of Eq. (3.96) and substituting the equilibrium relationship provided in Eq. (3.95) gives

$$\delta W = (F_1 \cos \alpha_1 - F_2 \cos \alpha_2 + F_3 \cos \alpha_3) \; \delta s = 0 \qquad (3.97)$$

Similarly, the virtual work is zero for the components along the *y* and *z* axes. In general, Eq. (3.97) requires

$$\delta W = 0 \qquad (3.98)$$

That is, virtual work must be equal to zero for a single particle in equilibrium under a set of forces.

In a rigid body, distances between particles remain constant, since no elongation or compression takes place under the action of forces. The virtual work done on each particle of the body when it is in equilibrium is zero. Hence the virtual work done by the entire rigid body is zero.

In general, then, for a rigid body in equilibrium, $\delta W = 0$.

Virtual Work on a Rigid Body. This principle of virtual work can be applied to idealized systems consisting of rigid elements. As an example, Fig. 3.59 shows a horizontal lever, which can be idealized as a rigid body. If a virtual rotation of $\delta\theta$ is applied, the virtual displacement for force $\boldsymbol{W}_1$ is $\boldsymbol{a \; \delta\theta}$, and for force $\boldsymbol{W}_2$, $\boldsymbol{b \; \delta\theta}$. Hence the virtual work during this rotation is

$$\delta W = W_1 a \; \delta\theta - W_2 b \; \delta\theta \qquad (3.99)$$

If the lever is in equilibrium, $\delta W = 0$. Hence $W_1 a = W_2 b$, which is the equilibrium condition that the sum of the moments of all forces about a support should be zero.

When the body is not rigid but can be distorted, the principle of virtual work as developed above cannot be applied directly. However, the principle can be modified to apply to bodies that undergo linear and nonlinear elastic deformations.

Strain Energy in a Bar. The internal work *U* done on elastic members is called **elastic potential energy,** or **strain energy.** Suppose, for example, that a bar (Fig. 3.60*a*) made of an elastic material, such as steel, is gradually elongated an amount Δ_f by a force P_f. As the

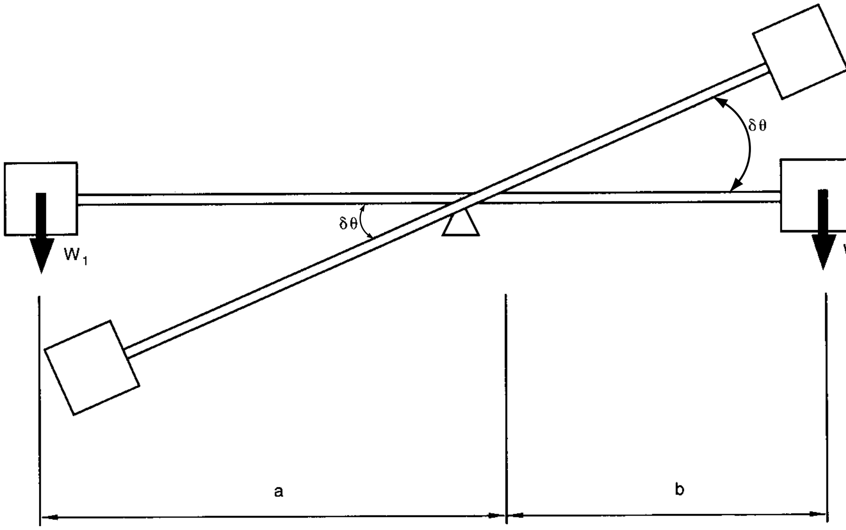


FIGURE 3.59 Virtual rotation of a lever.

bar stretches with increases in force from 0 to P_f, each increment of internal work dU may be expressed by Eq. (3.91) with $\alpha = 0$:

$$dU = P\,d\Delta \tag{3.100}$$

where $d\Delta$ = the current increment in displacement in the direction of P
P = the current applied force, $0 \leq P \leq P_f$

Equation (3.100) also may be written as

$$\frac{dU}{d\Delta} = P \tag{3.101}$$

which indicates that the derivative of the internal work with respect to a displacement (or

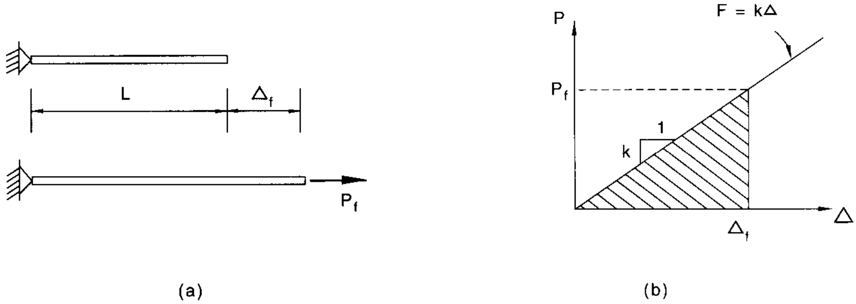


FIGURE 3.60 (*a*) Bar in tension elongates. (*b*) Energy stored in the bar is represented by the area under the load-displacement curve.

rotation) gives the corresponding force (or moment) at that location in the direction of the displacement (or rotation).

After the system comes to rest, a condition of equilibrium, the total internal work is

$$U = \int P\, d\Delta \tag{3.102}$$

The current displacement Δ is related to the applied force P by Eq. (3.51); that is, $P = EA\Delta/L$. Substitution into Eq. (3.102) yields

$$U = \int_0^{\Delta f} \frac{EA}{L} \Delta\, d\Delta = \frac{EA\delta_f^2}{2L} = \frac{LP_f^2}{2EA} = \frac{1}{2} P_f\, \Delta_f \tag{3.103}$$

When the force is plotted against displacement (Fig. 3.60*b*), the internal work is the shaded area under the line with slope $k = EA/L$.

When the bar in Fig. 3.60*a* is loaded and in equilibrium, the internal virtual work done by P_f during an additional virtual displacement $\delta\Delta$ equals the change in the strain energy of the bar:

$$\Delta U = k\Delta_f\, \delta\Delta \tag{3.104}$$

where Δ_f is the original displacement produced by P_f.

Principle of Virtual Work. This example illustrates the **principal of virtual work.** If an elastic body in equilibrium under the action of external loads is given a virtual deformation from its equilibrium condition, the work done by the external loads during this deformation equals the change in the internal work or strain energy, that is,

$$\delta W = \delta U \tag{3.105}$$

Hence, for the loaded bar in equilibrium (Fig. 3.60*a*), the external virtual work equals the internal virtual strain energy:

$$P_f\, \delta\Delta = k\Delta_f\, \delta\Delta \tag{3.106}$$

[For rigid bodies, no internal strain energy is generated, that is, $\delta U = k\Delta_f\, \delta\Delta = 0$, and Eq. (3.106) reduces the Eq. (3.98).] The example may be generalized to any constrained (supported) elastic body acted on by forces $P_1, P_2, P_3, \ldots$ for which the corresponding displacements are $\Delta_1, \Delta_2, \Delta_3, \ldots$. Equation (3.100) may then be expanded to

$$dU = \Sigma P_i\, d\Delta_i \tag{3.107}$$

Similarly, Eq. (3.101) may be generalized to

$$\frac{\partial U}{\partial \Delta_i} = P_i \tag{3.108}$$

The increase in strain energy due to the increments of the deformations is given by substitution of Eq. (3.108) into Eq. (3.107):

$$dU = \sum \frac{\partial U}{\partial \Delta_i} d\Delta_i = \frac{\partial U}{\partial \Delta_1} d\Delta_1 + \frac{\partial U}{\partial \Delta_2} d\Delta_2 + \frac{\partial U}{\partial \Delta_3} d\Delta_3 + \cdots \tag{3.109}$$

If specific deformations in Eq. (3.109) are represented by virtual displacements, load and deformation relationships for several structural systems may be obtained from the principle of virtual work.

Strain energy also can be generated when a member is subjected to other types of loads or deformations. The strain-energy equation can be written as a function of either load or deformation.

Strain Energy in Shear. For a member subjected to pure shear, strain energy is given by

$$U = \frac{V^2 L}{2AG} \tag{3.110a}$$

$$U = \frac{AG\Delta^2}{2L} \tag{3.110b}$$

where V = shear load
Δ = shear deformation
L = length over which the deformation takes place
A = shear area
G = shear modulus of elasticity

Strain Energy in Torsion. For a member subjected to torsion,

$$U = \frac{T^2 L}{2JG} \tag{3.111a}$$

$$U = \frac{JG\theta^2}{2L} \tag{3.111b}$$

where T = torque
θ = angle of twist
L = length over which the deformation takes place
J = polar moment of inertia
G = shear modulus of elasticity

Strain Energy in Bending. For a member subjected to pure bending (constant moment),

$$U = \frac{M^2 L}{2EI} \tag{3.112a}$$

$$U = \frac{EI\theta^2}{2L} \tag{3.112b}$$

where M = bending moment
θ = angle through which one end of beam rotates with respect to the other end
L = length over which the deformation takes place
I = moment of inertia
E = modulus of elasticity

For beams carrying transverse loads, the total strain energy is the sum of the energy for bending and that for shear.

Virtual Forces. Virtual work also may be created when a system of **virtual forces** is applied to a structure that is in equilibrium. In this case, the principle of virtual work requires that external virtual work, created by virtual forces acting over their induced displacements, equals the internal virtual work or strain energy. This concept is often used to determine deflections. For convenience, virtual forces are often represented by unit loads. Hence this method is frequently called the **unit-load method.**

Unit-Load Method. A unit load is applied at the location and in the direction of an unknown displacement Δ produced by given loads on a structure. According to the principle of virtual work, the external work done by the unit load equals the change in strain energy in the structure:

$$1\Delta = \Sigma fd \tag{3.113}$$

where Δ = deflection in desired direction produced by given loads
f = force in each element of the structure due to the unit load
d = deformation in each element produced by the given loads

The summation extends over all elements of the structure.

For a vertical component of a deflection, a unit vertical load should be used. For a horizontal component of a deflection, a unit horizontal load should be used. And for a rotation, a unit moment should be used.

For example, the deflection in the axial-loaded member shown in Fig. 3.60*a* can be determined by substituting $f = 1$ and $d = P_f L/EA$ into Eq. (3.113). Thus $1\Delta_f = 1P_f L/EA$ and $\Delta_f = P_f L/EA$.

For applications of the unit-load method for analysis of large structures, see Arts. 3.31 and 3.33.3.

(C. H. Norris et al., *Elementary Structural Analysis;* and R. C. Hibbeler, *Structural Analysis,* Prentice Hall, New Jersey.)

3.24 CASTIGLIANO'S THEOREMS

If strain energy U, as defined in Art. 3.23, is expressed as a function of external forces, the partial derivative of the strain energy with respect to one of the external forces P_i gives the displacement Δ_i corresponding to that force:

$$\frac{\partial U}{\partial P_i} = \Delta_i \tag{3.114}$$

This is known as **Castigliano's first theorem.**

If no displacement can occur at a support and Castigliano's theorem is applied to that support, Eq. (3.114) becomes

$$\frac{\partial U}{\partial P_i} = 0 \tag{3.115}$$

Equation (3.115) is commonly called the **principle of least work,** or **Castigliano's second theorem.** It implies that any reaction components in a structure will take on loads that will result in a minimum strain energy for the system. Castigliano's second theorem is restricted to linear elastic structures. On the other hand, Castigliano's first theorem is only limited to elastic structures and hence can be applied to nonlinear structures.

As an example, the principle of least work will be applied to determine the force in the vertical member of the truss shown in Fig. 3.61. If S_a denotes the force in the vertical bar, then vertical equilibrium requires the force in each of the inclined bars to be $(P - S_a)$ $(2 \cos \alpha)$. According to Eq. (3.103), the total strain energy in the system is

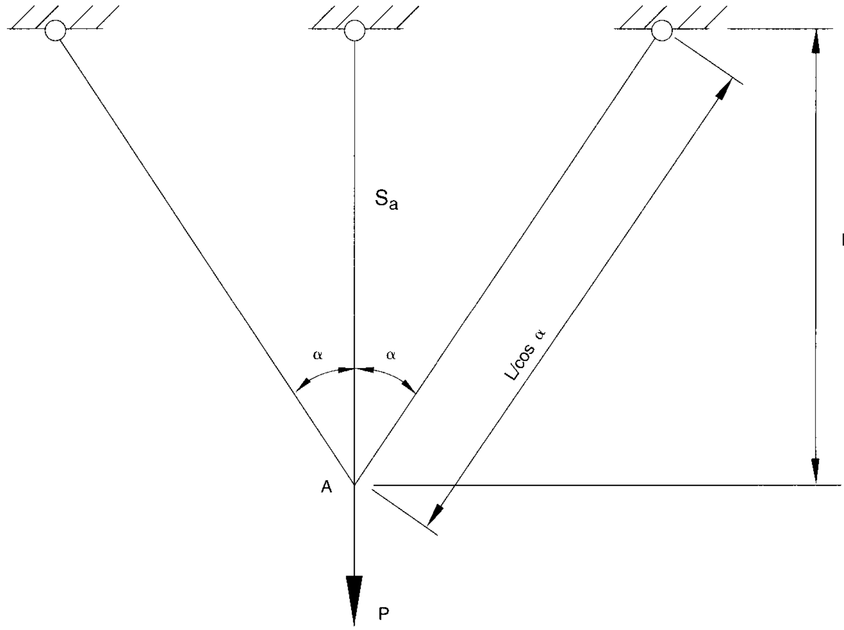


FIGURE 3.61 Statically indeterminate truss.

$$U = \frac{S_a^2 L}{2EA} + \frac{(P - S_a)^2 L}{4EA \cos^3 \alpha} \tag{3.116}$$

The internal work in the system will be minimum when

$$\frac{\partial U}{\partial S_a} = \frac{S_a L}{EA} - \frac{(P - S_a) L}{2EA \cos^3 \alpha} = 0 \tag{3.117a}$$

Solution of Eq. (3.117*a*) gives the force in the vertical bar as

$$S_a = \frac{P}{1 + 2 \cos^3 \alpha} \tag{3.117b}$$

(N. J. Hoff, *Analysis of Structures,* John Wiley & Sons, Inc., New York.)

3.25 RECIPROCAL THEOREMS

If the bar shown in Fig. 3.62*a*, which has a stiffness $k = EA/L$, is subjected to an axial force P_1, it will deflect $\Delta_1 = P_1/k$. According to Eq. (3.103), the external work done is $P_1\Delta_1/2$. If an additional load P_2 is then applied, it will deflect an additional amount $\Delta_2 = P_2/k$ (Fig. 3.62*b*). The additional external work generated is the sum of the work done in displacing P_1, which equals $P_1\Delta_2$, and the work done in displacing P_2, which equals $P_2\Delta_2/2$. The total external work done is

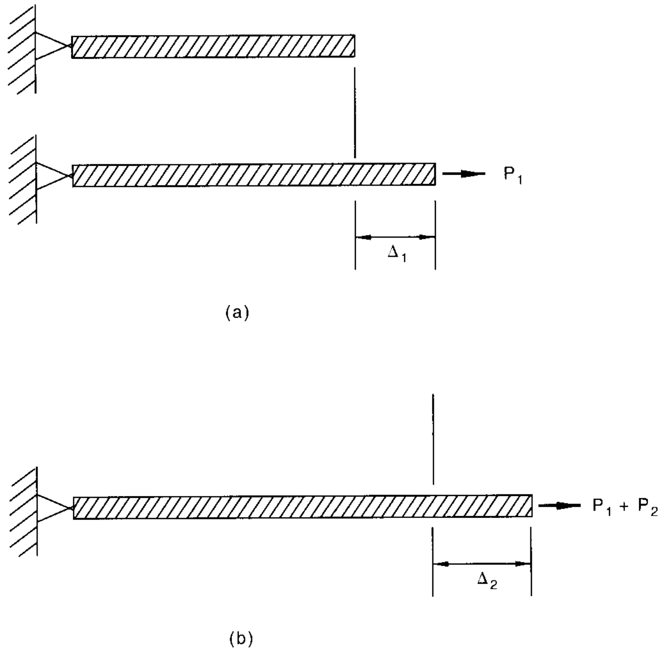


FIGURE 3.62 (*a*) Load on a bar performs work over displacement Δ_1. (*b*) Additional work is performed by both a second load and the original load.

$$W = \tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2 + P_1\Delta_2 \tag{3.118}$$

According to Eq. (3.103), the total internal work or strain energy is

$$U = \tfrac{1}{2}k\Delta_f^2 \tag{3.119}$$

where $\Delta_f = \Delta_1 + \Delta_2$. For the system to be in equilibrium, the total external work must equal the total internal work, that is

$$\tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2 + P_1\Delta_2 = \tfrac{1}{2}k\Delta_f^2 \tag{3.120}$$

If the bar is then unloaded and then reloaded by placing P_2 on the bar first and later applying P_1, the total external work done would be

$$W = \tfrac{1}{2}P_2\Delta_2 + \tfrac{1}{2}P_1\Delta_1 + P_2\Delta_1 \tag{3.121}$$

The total internal work should be the same as that for the first loading sequence because the total deflection of the system is still $\Delta_f = \Delta_1 + \Delta_2$. This implies that for a linear elastic system, the sequence of loading does not affect resulting deformations and corresponding internal forces. That is, in a **conservative system,** work is path-independent.

For the system to be in equilibrium under this loading, the total external work would again equal the total internal work:

$$\tfrac{1}{2}P_2\Delta_2 + \tfrac{1}{2}P_1\Delta_1 + P_2\Delta_1 = \tfrac{1}{2}\Delta_f^2 \tag{3.122}$$

Equating the left sides of Eqs. (3.120) and (3.122) and simplifying gives

$$P_1 \Delta_2 = P_2 \Delta_1 \tag{3.123}$$

This example, specifically Eq. (3.123), also demonstrates **Betti's theorem:** For a linearly elastic structure, the work done by a set of external forces P_1 acting through the set of displacements Δ_2 produced by another set of forces P_2 equals the work done by P_2 acting through the displacements Δ_1 produced by P_1.

Betti's theorem may be applied to a structure in which two loads P_i and P_j act at points i and j, respectively. P_i acting alone causes displacements Δ_{ii} and Δ_{ji}, where the first subscript indicates the point of displacement and the second indicates the point of loading. Application next of P_j to the system produces additional displacements Δ_{ij} and Δ_{jj}. According to Betti's theorem, for any P_i and P_j,

$$P_i \Delta_{ij} = P_j \Delta_{ji} \tag{3.124}$$

If $P_i = P_j$, then, according to Eq. (3.124), $\Delta_{ij} = \Delta_{ji}$. This relationship is known as **Maxwell's theorem of reciprocal displacements:** For a linear elastic structure, the displacement at point i due to a load applied at another point j equals the displacement at point j due to the same load applied at point i.

ANALYSIS OF STRUCTURAL SYSTEMS

A **structural system** consists of the primary load-bearing structure, including its members and connections. An analysis of a structural system consists of determining the reactions, deflections, and internal forces and corresponding stresses caused by external loads. Methods for determining these depend on both the external loading and the type of structural system that is assumed to resist these loads.

3.26 TYPES OF LOADS

Loads are forces that act or may act on a structure. For the purpose of predicting the resulting behavior of the structure, the loads, or external influences, including forces, consequent displacements, and support settlements, are presumed to be known. These influences may be specified by law, e.g., building codes, codes of recommended practice, or owner specifications, or they may be determined by engineering judgment. Loads are typically divided into two general classes: **dead load,** which is the weight of a structure including all of its permanent components, and **live load,** which is comprised of all loads other than dead loads.

The type of load has an appreciable influence on the behavior of the structure on which it acts. In accordance with this influence, loads may be classified as static, dynamic, long duration, or repetitive.

Static loads are those applied so slowly that the effect of time can be ignored. All structures are subject to some static loading, e.g., their own weight. There is, however, a large class of loads that usually is approximated by static loading for convenience. Occupancy loads and wind loads are often assumed static. All the analysis methods presented in the following articles, with the exception of Arts, 3.52 to 3.55, assume that static loads are applied to structures.

Dynamic loads are characterized by very short durations, and the response of the structure depends on time. Earthquake shocks, high-level wind gusts, and moving live loads belong in this category.

Long-duration loads are those which act on a structure for extended periods of time. For some materials and levels of stress, such loads cause structures to undergo deformations under constant load that may have serious effects. Creep and relaxation of structural materials may occur under long-duration loads. The weight of a structure and any superimposed dead load fall in this category.

Repetitive loads are those applied and removed many times. If repeated for a large number of times, they may cause the structure to fail in fatigue. Moving live load is in this category.

3.27 COMMONLY USED STRUCTURAL SYSTEMS

Structures are typically too complicated to analyze in their real form. To determine the response of a structure to external loads, it is convenient to convert the structural system to an idealized form. Stresses and displacements in trusses, for example, are analyzed based on the following assumptions.

3.27.1 Trusses

A **truss** is a structural system constructed of linear members forming triangular patterns. The members are assumed to be straight and connected to one another by frictionless hinges. All loading is assumed to be concentrated at these connections (joints or panel points). By virtue of these properties, truss members are subject only to axial load. In reality, these conditions may not be satisfied; for example, connections are never frictionless, and hence some moments may develop in adjoining members. In practice, however, assumption of the preceding conditions is reasonable.

If all the members are coplanar, then the system is called a **planar truss.** Otherwise, the structure is called a **space truss.** The exterior members of a truss are called **chords,** and the diagonals are called **web members.**

Trusses often act as beams. They may be constructed horizontally; examples include roof trusses and bridge trusses. They also may be constructed vertically; examples include transmission towers and internal lateral bracing systems for buildings or bridge towers and pylons. Trusses often can be built economically to span several hundred feet.

Roof trusses, in addition to their own weight, support the weight of roof sheathing, roof beams or purlins, wind loads, snow loads, suspended ceilings, and sometimes cranes and other mechanical equipment. Provisions for loads during construction and maintenance often need to be included. All applied loading should be distributed to the truss in such a way that the loads act at the joints. Figure 3.63 shows some common roof trusses.

Bridge trusses are typically constructed in pairs. If the roadway is at the level of the bottom chord, the truss is a **through truss.** If it is level with the top chord, it is a **deck truss.** The floor system consists of floor beams, which span in the transverse direction and connect to the truss joints; stringers, which span longitudinally and connect to the floor beams; and a roadway or deck, which is carried by the stringers. With this system, the dead load of the floor system and the bridge live loads it supports, including impact, are distributed to the truss joints. Figure 3.64 shows some common bridge trusses.

3.27.2 Rigid Frames

A **rigid frame** is a structural system constructed of members that resist bending moment, shear, and axial load and with connections that do not permit changes in the angles between the members under loads. Loading may be either distributed along the length of members, such as gravity loads, or entirely concentrated at the connections, such as wind loads.

If the axial load in a frame member is negligible, the member is commonly referred to as a **beam.** If moment and shear are negligible and the axial load is compressive, the member

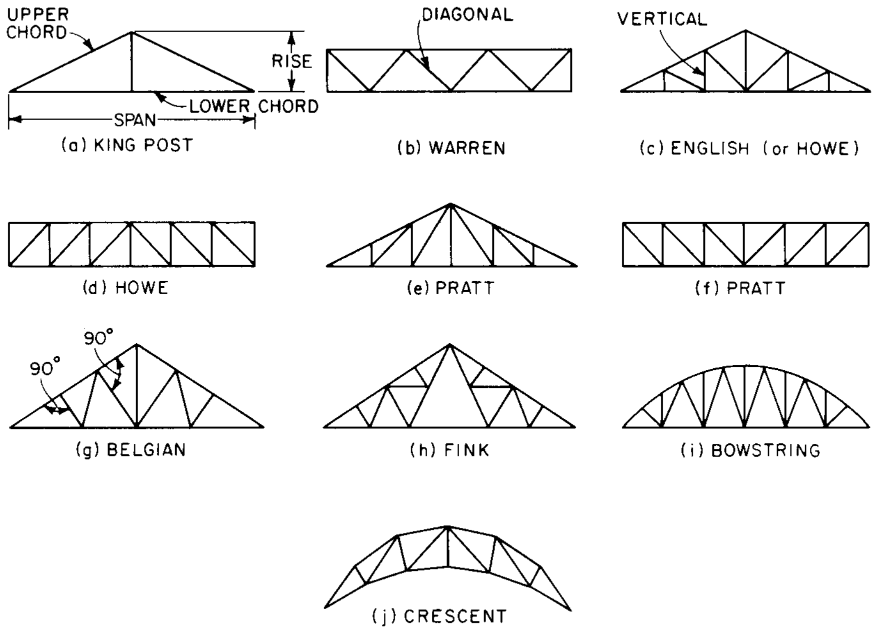


FIGURE 3.63 Common types of roof trusses.

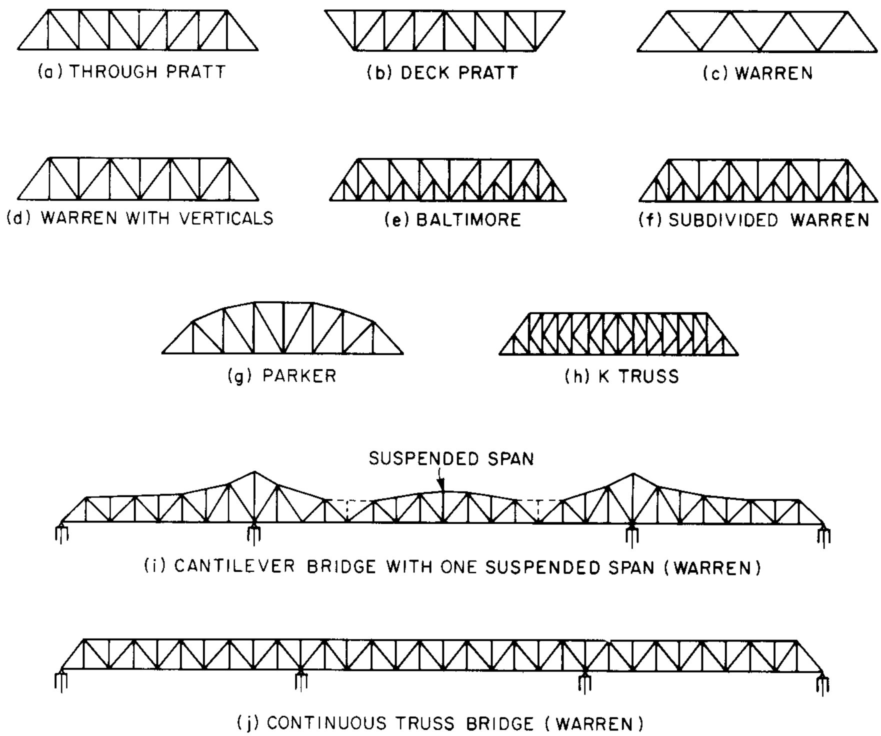


FIGURE 3.64 Common types of bridge trusses.

is referred to as a **column.** Members subjected to moments, shears, and compressive axial forces are typically called **beam-columns.** (Most vertical members are called **columns,** although technically they behave as beam-columns.)

If all the members are coplanar, the frame is called a **planar frame.** Otherwise, it is called a **space frame.** One plane of a space frame is called a **bent.** The area spanning between neighboring columns on a specific level is called a **bay.**

3.27.3 Continuous Beams

A **continuous beam** is a structural system that carries load over several spans by a series of rigidly connected members that resist bending moment and shear. The loading may be either concentrated or distributed along the lengths of members. The underlying structural system for many bridges is often a set of continuous beams.

3.28 DETERMINACY AND GEOMETRIC STABILITY

In a **statically determinate system,** all reactions and internal member forces can be calculated solely from equations of equilibrium. However, if equations of equilibrium alone do not provide enough information to calculate these forces, the system is **statically indeterminate.** In this case, adequate information for analyzing the system will only be gained by also considering the resulting structural deformations. Static determinacy is never a function of loading. In a statically determinate system, the distribution of internal forces is not a function of member cross section or material properties.

In general, the degree of static determinacy n for a truss may be determined by

$$n = m - \alpha j + R \tag{3.125}$$

where m = number of members
j = number of joints including supportjs
α = dimension of truss ($\alpha = 2$ for a planar truss and $\alpha = 3$ for a space truss)
R = number of reaction components

Similarly, the degree of static determinacy for a frame is given by

$$n = 3(\alpha - 1)(m - j) + R \tag{3.126}$$

where $\alpha = 2$ for a planar frame and $\alpha = 3$ for a space frame.

If n is greater than zero, the system is geometrically stable and statically indeterminate; if n is equal to zero, it is statically determinate and may or may not be stable; if n is less than zero, it is always geometrically unstable. Geometric instability of a statically determinate truss ($n = 0$) may be determined by observing that multiple solutions to the internal forces exist when applying equations of equilibrium.

Figure 3.65 provides several examples of statically determinate and indeterminate systems. In some cases, such as the planar frame shown in Fig. 3.65*e*, the frame is statically indeterminate for computation of internal forces, but the reactions can be determined from equilibrium equations.

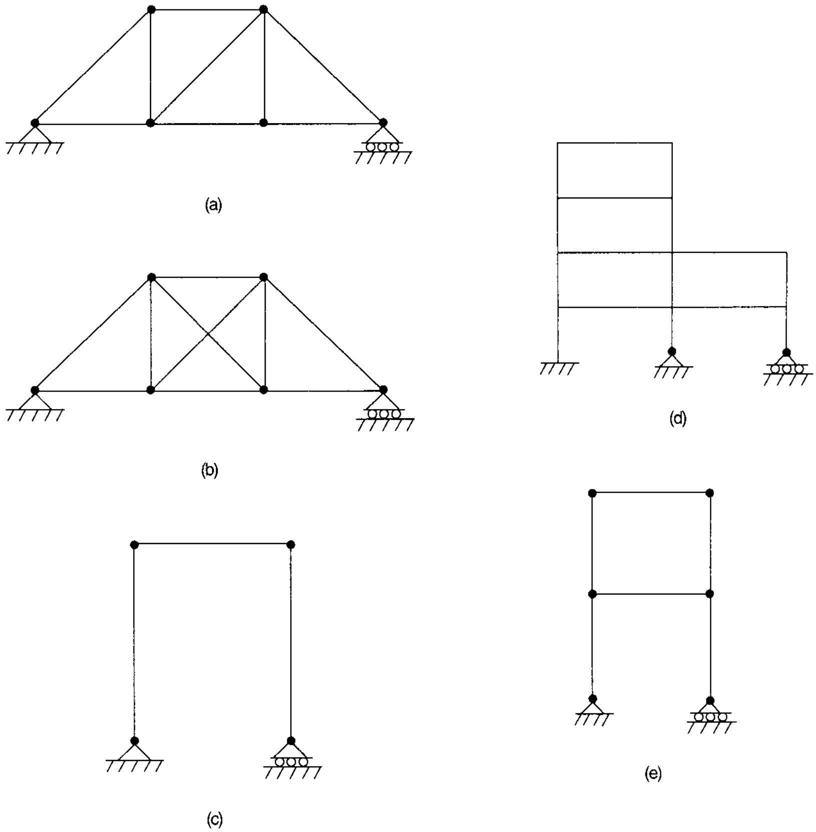


FIGURE 3.65 Examples of statically determinate and indeterminate systems: (*a*) Statically determinate truss ($n = 0$); (*b*) statically indeterminate truss ($n = 1$); (*c*) statically determinate frame ($n = 0$); (*d*) statically indeterminate frame ($n = 15$); (*e*) statically indeterminate frame ($n = 3$).

3.29 CALCULATION OF REACTIONS IN STATICALLY DETERMINATE SYSTEMS

For statically determinate systems, reactions can be determined from equilibrium equations [Eq. (3.11) or (3.12)]. For example, in the planar system shown in Fig. 3.66, reactions R_1, H_1, and R_2 can be calculated from the three equilibrium equations. The beam with overhang carries a uniform load of 3 kips/ft over its 40-ft horizontal length, a vertical 60-kip concentrated load at *C,* and a horizontal 10-kip concentrated load at *D.* Support *A* is hinged; it can resist vertical and horizontal forces. Support *B*, 30 ft away, is on rollers; it can resist only vertical force. Dimensions of the member cross sections are assumed to be small relative to the spans.

Only support *A* can resist horizontal loads. Since the sum of the horizontal forces must equal zero and there is a 10-kip horizontal load at *D*, the horizontal component of the reaction at *A* is $H_1 = 10$ kips.

The vertical reaction at *A* can be computed by setting the sum of the moments of all forces about *B* equal to zero:

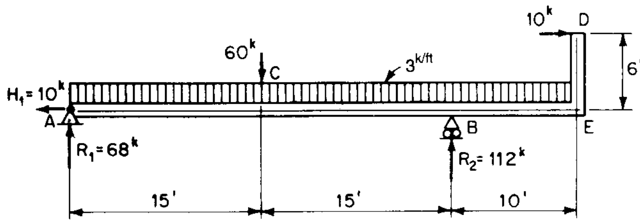


FIGURE 3.66 Beam with overhang with uniform and concentrated loads.

$$3 \times 40 \times 10 + 60 \times 15 - 10 \times 6 - 30R_1 = 0$$

from which $R_1 = 68$ kips. Similarly, the reaction at B can be found by setting the sum of the moments about A of all forces equal to zero:

$$3 \times 40 \times 20 + 60 \times 15 + 10 \times 6 - 30R_2 = 0$$

from which $R_2 = 112$ kips. Alternatively, equilibrium of vertical forces can be used to obtain R_2, given $R_1 = 68$:

$$R_2 + R_1 - 3 \times 40 - 60 = 0$$

Solution of this equation also yields $R_2 = 112$ kips.

3.30 FORCES IN STATICALLY DETERMINATE TRUSSES

A convenient method for determining the member forces in a truss is to isolate a portion of the truss. A section should be chosen such that it is possible to determine the forces in the cut members with the equations of equilibrium [Eq. (3. 11) or (3.12)]. Compressive forces act toward the panel point, and tensile forces act away from the panel point.

3.30.1 Method of Sections

To calculate the force in member *a* of the truss in Fig. 3.67*a*, the portion of the truss in Fig. 3.67*b* is isolated by passing section *x*–*x* through members *a*, *b*, and *c*. Equilibrium of this part of the truss is maintained by the 10-kip loads at panel points U_1 and U_2, the 25-kip reaction, and the forces S_a, S_b, and S_c in members *a*, *b*, and *c*, respectively. S_a can be determined by equating to zero the sum of the moments of all the external forces about panel point L_3, because the other unknown forces S_b and S_c pass through L_3 and their moments therefore equal zero. The corresponding equilibrium equation is

$$-9S_a + 36 \times 25 - 24 \times 10 - 12 \times 10 = 0$$

Solution of this equation yields $S_a = 60$ kips. Similarly, S_b can be calculated by equating to zero the sum of the moments of all external forces about panel point U_2:

$$-9S_b + 24 \times 25 - 12 \times 10 = 0$$

for which $S_b = 53.3$ kips.

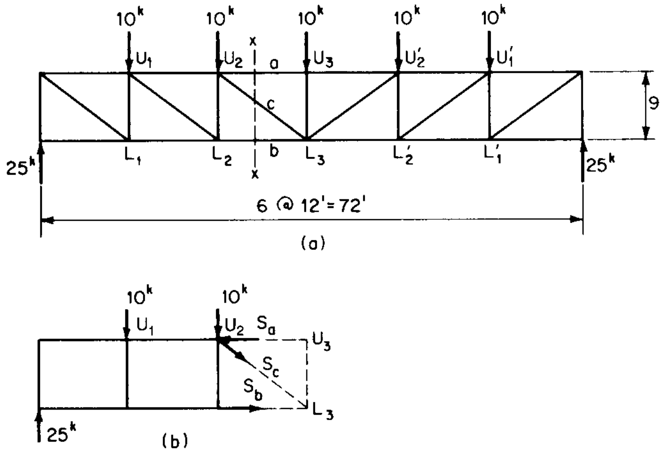


FIGURE 3.67 (*a*) Truss with loads at panel points. (*b*) Stresses in members cut by section *x*—*x* hold truss in equilibrium.

Since members *a* and *b* are horizontal, they do not have a vertical component. Hence diagonal *c* must carry the entire vertical shear on section *x–x*: 25 − 10 − 10 = 5 kips. With 5 kips as its vertical component and a length of 15 ft on a rise of 9 ft,

$$S_c = 15/9 \times 5 = 8.3 \text{ kips}$$

When the chords are not horizontal, the vertical component of the diagonal may be found by subtracting from the shear in the section the vertical components of force in the chords.

3.30.2 Method of Joints

A special case of the method of sections is choice of sections that isolate the joints. With the forces in the cut members considered as external forces, the sum of the horizontal components and the sum of the vertical components of the external forces acting at each joint must equal zero.

Since only two equilibrium equations are available for each joint, the procedure is to start with a joint that has two or fewer unknowns (usually a support). When these unknowns have been found, the procedure is repeated at successive joints with no more than two unknowns.

For example, for the truss in Fig. 3.68*a*, at joint 1 there are three forces: the reaction of 12 kips, force S_a in member *a*, and force S_c in member *c*. Since *c* is horizontal, equilibrium of vertical forces requires that the vertical component of force in member *a* be 12 kips. From the geometry of the truss, $S_a = 12 - 15/9 = 20$ kips. The horizontal component of S_a is $20 \times 12/15 = 16$ kips. Since the sum of the horizontal components of all forces acting at joint 1 must equal zero, $S_c = 16$ kips.

At joint 2, the force in member *e* is zero because no vertical forces are present there. Hence, the force in member *d* equals the previously calculated 16-kip force in member *c*. Forces in the other members would be determined in the same way (see Fig. 3.68*d*, *e*, and *f*).

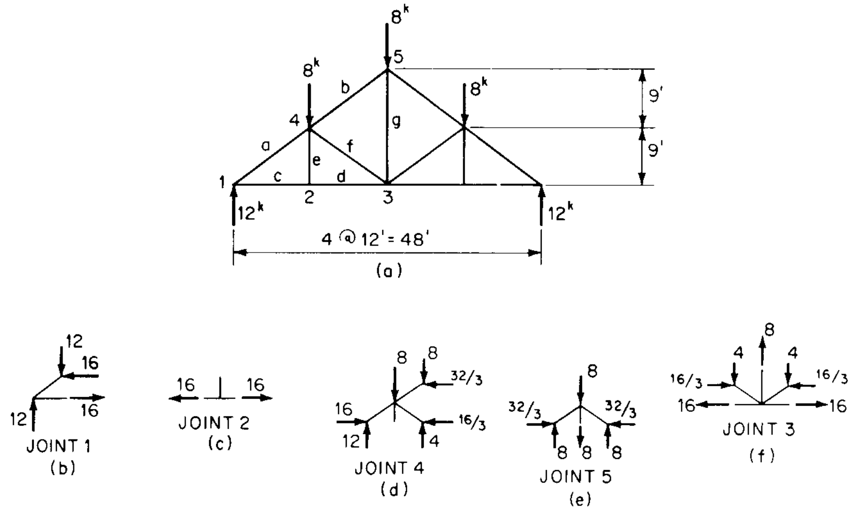


FIGURE 3.68 Calculation of truss stresses by method of joints.

3.31 DEFLECTIONS OF STATICALLY DETERMINATE TRUSSES

In Art. 3.23, the basic concepts of virtual work and specifically the unit-load method are presented. Employing these concepts, this method may be adapted readily to computing the deflection at any panel point (joint) in a truss.

Specifically, Eq. (3.113), which equates external virtual work done by a virtual unit load to the corresponding internal virtual work, may be written for a truss as

$$1\Delta = \sum_{i=1}^{n} f_i \frac{P_i L_i}{E_i A_i} \tag{3.127}$$

where Δ = displacement component to be calculated (also the displacement at and in the direction of an applied unit load)

n = total number of members

f_i = axial force in member i due to unit load applied at and in the direction of the desired Δ—horizontal or vertical unit load for horizontal or vertical displacement, moment for rotation

P_i = axial force in member i due to the given loads

L_i = length of member i

E_i = modulus of elasticity for member i

A_i = cross-sectional area of member i

To find the deflection Δ at any joint in a truss, each member force P_i resulting from the given loads is first calculated. Then each member force f_i resulting from a unit load being applied at the joint where Δ occurs and in the direction of Δ is calculated. If the structure is statically determinate, both sets of member forces may be calculated from the method of joints (Sec. 3.30.2). Substituting each member's forces P_i and f_i and properties L_i, E_i, and A_i, into Eq. (3.127) yields the desired deflection Δ.

As an example, the midspan downward deflection for the truss shown in Fig. 3.68*a* will be calculated. The member forces due to the 8-kip loads are shown in Fig. 3.69*a*. A unit load acting downward is applied at midspan (Fig. 3.69*b*). The member forces due to the unit

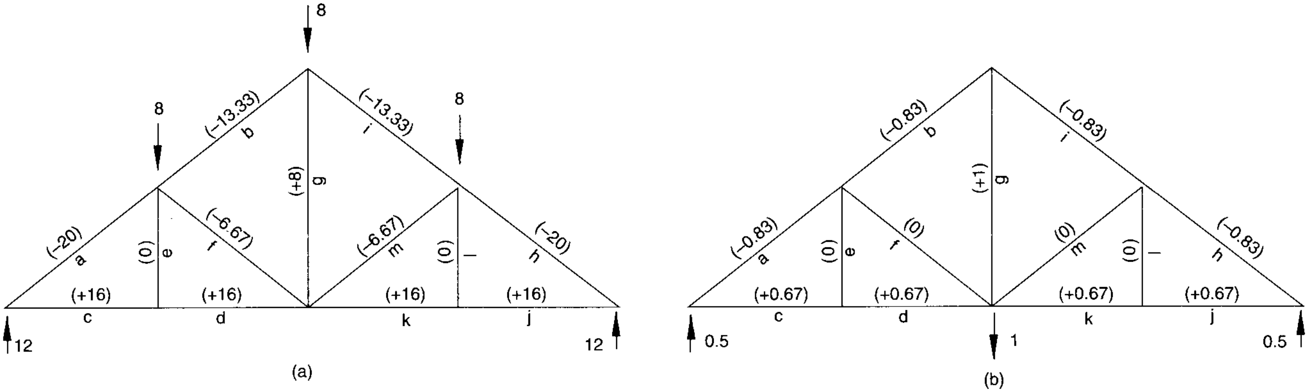


FIGURE 3.69 (*a*) Loaded truss with stresses in members shown in parentheses. (*b*) Stresses in truss due to a unit load applied for calculation of midspan deflection.

load are shown in Fig. 3.69*b*. On the assumption that all members have area $A_i = 2$ in^2 and modulus of elasticity $E_i = 29{,}000$ ksi, Table 3.3 presents the computations for the midspan deflection Δ. Members not stressed by either the given loads, $P_i = 0$, or the unit load, $f_i = 0$, are not included in the table. The resulting midspan deflection is calculated as 0.31 in.

3.32 FORCES IN STATICALLY DETERMINATE BEAMS AND FRAMES

Similar to the method of sections for trusses discussed in Art. 3.30, internal forces in statically determinate beams and frames also may be found by isolating a portion of these systems. A section should be chosen so that it will be possible to determine the unknown internal forces from only equations of equilibrium [Eq. (3.11) or 3.12)].

As an example, suppose that the forces and moments at point *A* in the roof purlin of the **gable frame** shown in Fig. 3.70*a* are to be calculated. Support *B* is a hinge. Support *C* is on rollers. Support reactions R_1, H_1, and R_2 are determined from equations of equilibrium. For example, summing moments about *B* yields

$$\Sigma M = 30 \times R_2 + 12 \times 8 - 15 \times 12 - 30 \times 6 = 0$$

from which $R_2 = 8.8$ kips. $R_1 = 6 + 12 + 6 - 8.8 = 15.2$ kips.

The portion of the frame shown in Fig. 3.70*b* is then isolated. The internal shear V_A is assumed normal to the longitudinal axis of the rafter and acting downward. The axial force P_A is assumed to cause tension in the rafter. Equilibrium of moments about point *A* yields

$$\Sigma M = M_A + 10 \times 6 + (12 + 10 \tan 30) \times 8 - 10 \times 15.2 = 0$$

from which $M_A = -50.19$ kips-ft. Vertical equilibrium of this part of the frame is maintained with

$$\Sigma F_y = 15.2 - 6 + P_A \sin 30 - V_A \cos 30 = 0 \qquad (3.128)$$

Horizontal equilibrium requires that

TABLE 3.3 Calculation of Truss Deflections

Member	P_i, kips	f_i	$\dfrac{1000\,L_i}{E_iA_i}$	$f_i\,\dfrac{P_iL_i}{E_iA_i}$, in
a	−20.00	−0.83	3.103	0.052
b	−13.33	−0.83	3.103	0.034
c	16.00	0.67	2.483	0.026
d	16.00	0.67	2.483	0.026
g	8.00	1.00	3.724	0.030
h	−20.00	−0.83	3.103	0.052
i	−13.33	−0.83	3.103	0.034
j	16.00	0.67	2.483	0.026
k	16.00	0.67	2.483	0.026
				$\Delta = \Sigma = 0.306$

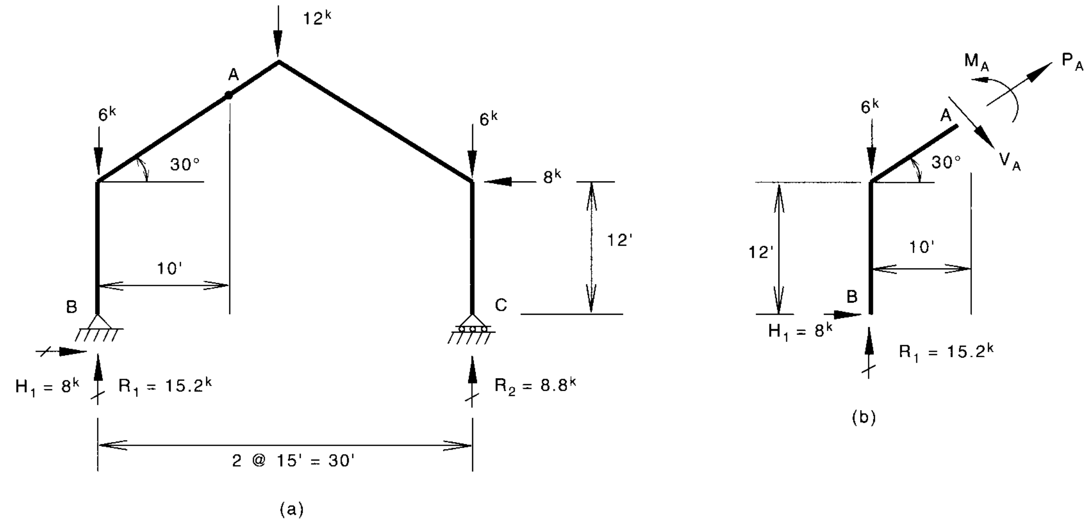


FIGURE 3.70 (*a*) Loaded gable frame. (*b*) Internal forces hold portion of frame in equilibrium.

$$\Sigma F_x = 8 + P_A \cos 30 + V_A \sin 30 = 0 \tag{3.129}$$

Simultaneous solution of Eqs. (3.128) and (3.129) gives $V_A = 3.96$ hips and $P_A = -11.53$ kips. The negative value indicates that the rafter is in compression.

3.33 DEFORMATIONS IN BEAMS

Article 3.18 presents relationships between a distributed load on a beam, the resulting internal forces and moments, and the corresponding deformations. These relationships provide the key expressions used in the **conjugate-beam method** and the **moment-area method** for computing beam deflections and slopes of the neutral axis under loads. The unit-load method used for this purpose is derived from the principle of virtual work (Art. 3.23).

3.33.1 Conjugate-Beam Method

For a beam subjected to a distributed load $w(x)$, the following integral relationships hold:

$$V(x) = \int w(x)\, dx \tag{3.130a}$$

$$M(x) = \int V(x)\, dx = \int\int w(x)\, dx\, dx \tag{3.130b}$$

$$\theta(x) = \int \frac{M(x)}{EI(x)}\, dx \tag{3.130c}$$

$$\delta(x) = \int \theta(x)\, dx = \int\int \frac{M(x)}{EI(x)}\, dx\, dx \tag{3.130d}$$

Comparison of Eqs. (3.130*a*) and (3.130*b*) with Eqs. (3.130*c*) and (3.130*d*) indicates that for a beam subjected to a distributed load $w(x)$, the resulting slope $\theta(x)$ and deflection $\delta(x)$ are equal, respectively, to the corresponding shear distribution $\overline{V}(x)$ and moment distribution $\overline{M}(x)$ generated in an associated or conjugated beam subjected to the distributed load $M(x)/EI(x)$. $M(x)$ is the moment at x due to the actual load $w(x)$ on the original beam.

In some cases, the supports of the real beam should be replaced by different supports for the conjugate beam to maintain the consistent θ-to-$\overline{V}$ and δ-to-$\overline{M}$ correspondence. For example, at the fixed end of a cantilevered beam, there is no rotation ($\theta = 0$) and no deflection ($\delta = 0$). Hence, at this location in the conjugate beam, $\overline{V} = 0$ and $\overline{M} = 0$. This can only be accomplished with a free-end support; i.e., a fixed end in a real beam is represented by a free end in its conjugate beam. A summary of the corresponding support conditions for several conjugate beams is provided in Fig. 3.71.

The sign convention to be employed for the conjugate-beam method is as follows:

A positive M/EI segment in the real beam should be placed as an upward (positive) distributed load $\overline{w}$ on the conjugate beam. A negative M/EI segment should be applied as a downward (negative) $\overline{w}$.

Positive shear $\overline{V}$ in the conjugate beam corresponds to a counterclockwise (positive) slope θ in the real beam. Negative $\overline{V}$ corresponds to a clockwise (negative) θ.

Positive moment $\overline{M}$ in the conjugate beam corresponds to an upward (positive) deflection δ in the real beam. Negative $\overline{M}$ corresponds to downward (positive) δ.

As an example, suppose the deflection at point B in the cantilevered beam shown in Fig. 3.72*a* is to be calculated. With no distributed load between the tip of the beam and its support, the bending moments on the beam are given by $M(x) = P(x - L)$ (Fig. 3.72*b*). The conjugate beam is shown in Fig. 3.72*c*. It has the same physical dimensions (E, I, and L) as the original beam but interchanged support conditions and is subject to a distributed load $\overline{w}(x) = P(x - L)/EI$, as indicated in Fig. 3.72*c*. Equilibrium of the free-body diagram shown in Fig. 3.73*d* requires $\overline{V}_B = -15PL^2/32EI$ and $\overline{M}_B = -27PL^3/128EI$. The slope in the real beam at point B is then equal to the conjugate shear at this point, $\theta_B = \overline{V}_B = -15PL^2/32EI$. Similarly, the deflection at point B is the conjugate moment, $\delta_B = \overline{M}_B = -27PL^3/128EI$. See also Sec. 3.33.2.

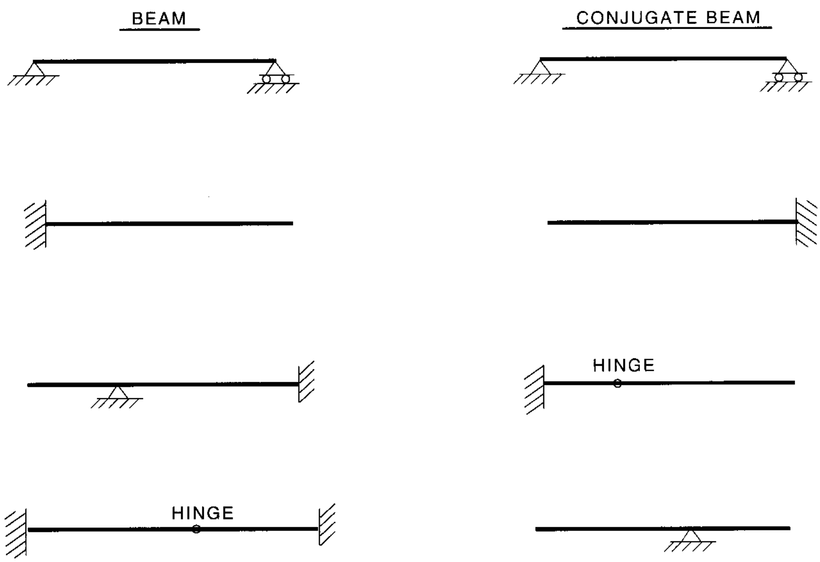


FIGURE 3.71 Beams and corresponding conjugate beams for various types of supports.

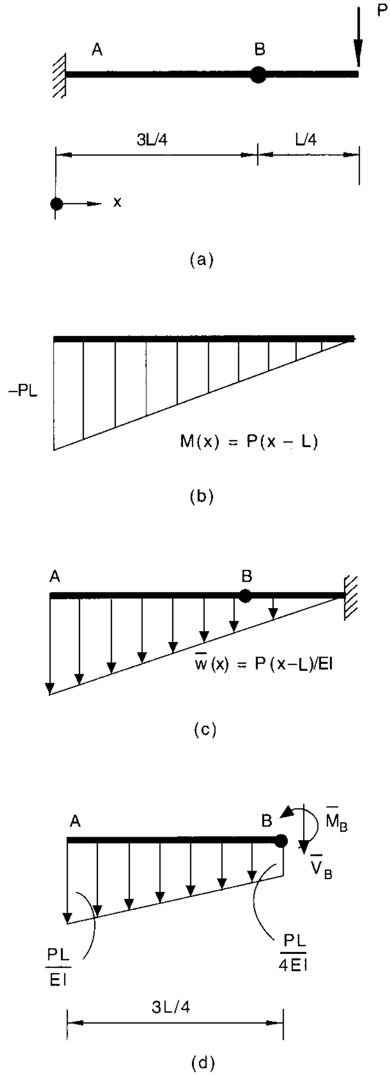


FIGURE 3.72 Deflection calculations for a cantilever by the conjugate beam method. (*a*) Cantilever beam with a load on the end. (*b*) Bending-moment diagram. (*c*) Conjugate beam loaded with M/EI distribution. (*d*) Deflection at B equals the bending moment at B due to the M/EI loading.

3.33.2 Moment-Area Method

Similar to the conjugate-beam method, the moment-area method is based on Eqs. (3.130*a*) to (3.130*d*). It expresses the deviation in the slope and tangential deflection between points *A* and *B* on a deflected beam:

$$\theta_B - \theta_A = \int_{x_A}^{x_B} \frac{M(x)}{EI(x)}\, dx \tag{3.131a}$$

$$t_B - t_A = \int_{x_A}^{x_B} \frac{M(x)x}{EI(x)}\, dx \tag{3.131b}$$

Equation (3.131*a*) indicates that the change in slope of the elastic curve of a beam between any two points equals the area under the *M/EI* diagram between these points. Similarly, Eq. (3.131*b*) indicates that the tangential deviation of any point on the elastic curve with respect to the tangent to the elastic curve at a second point equals the moment of the area under the *M/EI* diagram between the two points taken about the first point.

For example, deflection δ_B and rotation θ_B at point *B* in the cantilever shown in Fig. 3.72*a* are

$$\begin{aligned}
\theta_B &= \theta_A + \int_0^{3L/4} \frac{M(x)}{EI}\, dx \\
&= 0 + \left(-\frac{PL}{4EI}\frac{3L}{4} - \frac{1}{2}\frac{3PL}{4EI}\frac{3L}{4} \right) \\
&= -\frac{15PL^2}{32EI}
\end{aligned}$$

$$\begin{aligned}
t_B &= t_A + \int_0^{3L/4} \frac{M(x)x}{EI}\, dx \\
&= 0 + \left(-\frac{PL}{4EI}\frac{3L}{4}\frac{1}{2}\frac{3L}{4} - \frac{1}{2}\frac{3PL}{4EI}\frac{3L}{4}\frac{2}{3}\frac{3L}{4} \right) \\
&= -\frac{27PL^3}{128EI}
\end{aligned}$$

For this particular example $t_A = 0$, and hence $\delta_B = t_B$.

The moment-area method is particularly useful when a point of zero slope can be identified. In cases where a point of zero slope cannot be located, deformations may be more readily calculated with the conjugate-beam method. As long as the bending-moment diagram can be defined accurately, both methods can be used to calculate deformations in either statically determinate or indeterminate beams.

3.33.3 Unit-Load Method

Article 3.23 presents the basic concepts of the unit-load method. Article 3.31 employs this method to compute the deflections of a truss. The method also can be adapted to compute deflections in beams.

The deflection Δ at any point of a beam due to bending can be determined by transforming Eq. (3.113) to

$$1\Delta = \int_0^L \frac{M(x)}{EI(x)} m(x)\, dx \tag{3.132}$$

where $M(x)$ = moment distribution along the span due to the given loads
E = modulus of elasticity
I = cross-sectional moment of inertia
L = beam span
$m(x)$ = bending-moment distribuiton due to a unit load at the location and in the direction of deflection Δ

As an example of the use of Eq. (3.132), the midspan deflection will be determined for a prismatic, simply supported beam under a uniform load w (Fig. 3.73*a*). With support A as the origin, the equation for bending movement due to the uniform load is $M(x) = wLx/2 - wx^2/2$ (Fig. 3.73*b*). For a unit vertical load at midspan (Fig. 3.73*c*), the equation for bending moment in the left half of the beam is $m(x) = x/2$ and in the right $m(x) = (L - x)/2$ (Fig. 3.73*d*). By Eq. (3.132), the deflection is

$$\Delta = \frac{1}{EI}\int_0^{L/2} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{x}{2}\, dx + \frac{1}{EI}\int_{L/2}^{L} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{L-x}{2}\, dx$$

from which $\Delta = 5wL^4/384EI$. If the beam were not prismatic, EI would be a function of x and would be inside the integral.

Equation (3.113) also can be used to calculate the slope at any point along a beam span. Figure 3.74*a* shows a simply supported beam subjected to a moment M_A acting at support A. The resulting moment distribution is $M(x) = M_A\,(1 - x/L)$ (Fig. 3.74*b*). Suppose that the rotation θ_B at support B is to be determined. Application of a unit moment at B (Fig. 3.74*c*) results in the moment distribution $m(x) = x/L$ (Fig. 3.74*d*). By Eq. (3.132), on substitution of θ_B for Δ, the rotation at B is

$$\theta_B = \int_0^L \frac{M(x)}{EI(x)} m(x)\, dx = \frac{M_A}{EI}\int_0^L \left(1 - \frac{x}{L}\right)\frac{x}{L}\, dx = \frac{M_A L}{6EI}$$

(C. H. Norris et al., *Elementary Structural Analysis,* McGraw-Hill, Inc., New York; J. McCormac and R. E. Elling, *Structural Analysis—A Classical and Matrix Approach,* Harper and Row Publishers, New York; R. C. Hibbeler, *Structural Analysis,* Prentice Hall, New Jersey.)

3.34 METHODS FOR ANALYSIS OF STATICALLY INDETERMINATE SYSTEMS

For a statically indeterminate structure, equations of equilibrium alone are not sufficient to permit analysis (see Art. 3.28). For such systems, additional equations must be derived from requirements ensuring compatibility of deformations. The relationship between stress and strain affects compatibility requirements. In Arts. 3.35 to 3.39, linear elastic behavior is assumed; i.e., in all cases stress is assumed to be directly proportional to strain.

There are two basic approaches for analyzing statically indeterminate structures, force methods and displacement methods. In the **force methods,** forces are chosen as redundants to satisfy equilibrium. They are determined from compatibility conditions (see Art. 3.35). In the **displacement methods,** displacements are chosen as redundants to ensure geometric compatibility. They are also determined from equilibrium equations (see Art. 3.36). In both

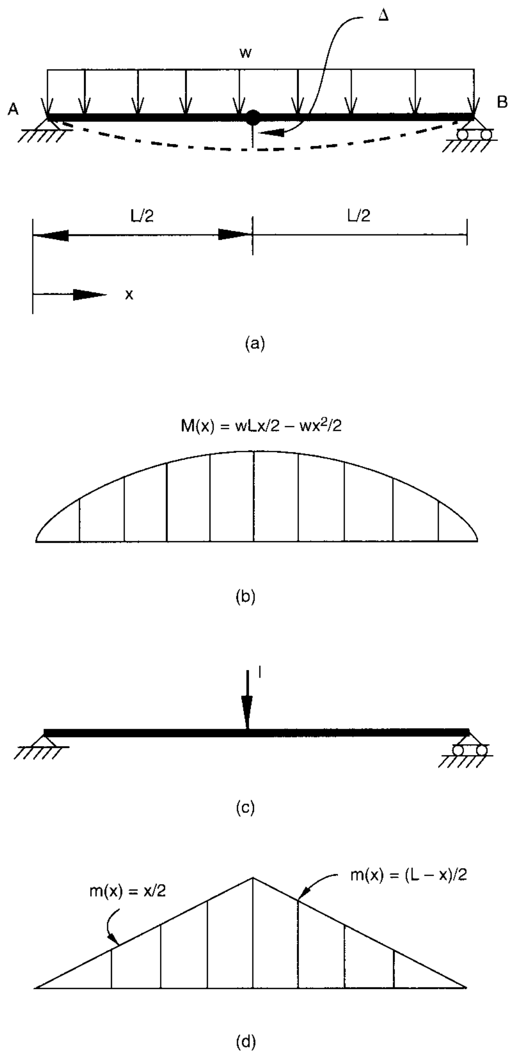


FIGURE 3.73 Deflection calculations for a simple beam by unit load method. (*a*) Uniformly loaded beam. (*b*) Bending-moment diagram for the uniform load. (*c*) Unit load at midspan. (*d*) Bending-moment diagram for the unit load.

methods, once the unknown redundants are determined, the structure can be analyzed by statics.

3.35 FORCE METHOD (METHOD OF CONSISTENT DEFLECTIONS)

For analysis of a statically indeterminate structure by the force method, the degree of indeterminacy (number of redundants) n should first be determined (see Art. 3.28). Next, the

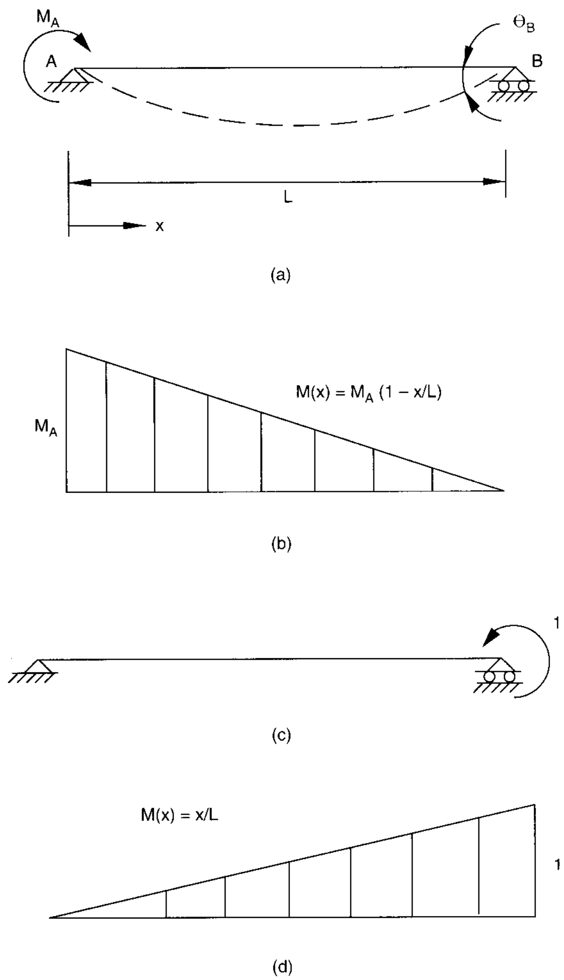


FIGURE 3.74 Calculation of end rotations of a simple beam by the unit-load method. (*a*) Moment applied at one end. (*b*) Bending-moment diagram for the applied moment. (*c*) Unit load applied at end where rotation is to be determined. (*d*) Bending-moment diagram for the unit load.

structure should be reduced to a statically determinate structure by release of n constraints or redundant forces (X_1, X_2, X_3, . . . , X_n). Equations for determination of the redundants may then be derived from the requirements that equilibrium must be maintained in the reduced structure and deformations should be compatible with those of the original structure.

Displacements δ_1, δ_2, δ_3, . . . ,δ_n in the reduced structure at the released constraints are calculated for the original loads on the structure. Next, a separate analysis is performed for each released constraint j to determine the displacements at all the released constraints for a unit load applied at j in the direction of the constraint. The displacement f_{ij} at constraint i due to a unit load at released constraint j is called a **flexibility coefficient.**

Next, displacement compatibility at each released constraint is enforced. For any constraint i, the displacement δ_i due to the given loading on the reduced structure and the sum of the displacements $f_{ij}X_j$ in the reduced structure caused by the redundant forces are set equal to known displacement Δ_i of the original structure:

$$\Delta_i = \delta_i + \sum_{j=1}^{n} f_{ij}X_j \qquad i = 1, 2, 3, \ldots, n \tag{3.133}$$

If the redundant i is a support that has no displacement, then $\Delta_i = 0$. Otherwise, Δ_i will be a known support displacement. With n constraints, Eq. (3.133) provides n equations for solution of the n unknown redundant forces.

As an example, the continuous beam shown in Fig. 3.75*a* will be analyzed. If axial-force effects are neglected, the beam is indeterminate to the second degree ($n = 2$). Hence two redundants should be chosen for removal to obtain a statically determinate (reduced) structure. For this purpose, the reactions R_B at support B and R_C at support C are selected. Displacements of the reduced structure may then be determined by any of the methods presented in Art. 3.33. Under the loading shown in Fig. 3.75*a*, the deflections at the redundants are $\delta_B = -5.395$ in and $\delta_C = -20.933$ in (Fig. 3.75*b*). Application of an upward-acting unit load to the reduced beam at B results in deflections $f_{BB} = 0.0993$ in at B and $f_{CB} = 0.3228$ in at C (Fig. 3.75*c*). Similarly, application of an upward-acting unit load at C results in $f_{BC} = 0.3228$ in at B and $f_{CC} = 1.3283$ in at C (Fig. 3.75*d*). Since deflections cannot occur at supports B and C, Eq. (3.133) provides two equations for displacement compatibility at these supports:

$$0 = -5.3955 + 0.0993R_B + 0.3228R_C$$

$$0 = -20.933 + 0.3228R_B + 1.3283R_C$$

Solution of these simultaneous equations yields $R_B = 14.77$ kips and $R_C = 12.17$ kips. With these two redundants known, equilibrium equations may be used to determine the remaining reactions as well as to draw the shear and moment diagrams (see Art. 3.18 and 3.32).

In the preceding example, in accordance with the reciprocal theorem (Art. 3.22), the flexibility coefficients f_{CB} and f_{BC} are equal. In linear elastic structures, the displacement at constraint i due to a load at constraint j equals the displacement at constraint j when the same load is applied at constraint i; that is, $f_{ij} = f_{ji}$. Use of this relationship can significantly reduce the number of displacement calculations needed in the force method.

The force method also may be applied to statically indeterminate trusses and frames. In all cases, the general approach is the same.

(F. Arbabi, *Structural Analysis and Behavior,* McGraw-Hill, Inc., New York; J. McCormac and R. E. Elling, *Structural Analysis—A Classicial and Matrix Approach,* Harper and Row Publishers, New York; and R. C. Hibbeler, *Structural Analysis,* Prentice Hall, New Jersey.)

3.36 DISPLACEMENT METHODS

For analysis of a statically determinate or indeterminate structure by any of the displacement methods, independent displacements of the joints, or **nodes,** are chosen as the unknowns. If the structure is defined in a three-dimensional, orthogonal coordinate system, each of the three translational and three rotational displacement components for a specific node is called a **degree of freedom.** The displacement associated with each degree of freedom is related to corresponding deformations of members meeting at a node so as to ensure geometric compatibility.

Equilibrium equations relate the unknown displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ at degrees of freedom 1, 2, . . . , *n,* respectively, to the loads P_i on these degrees of freedom in the form

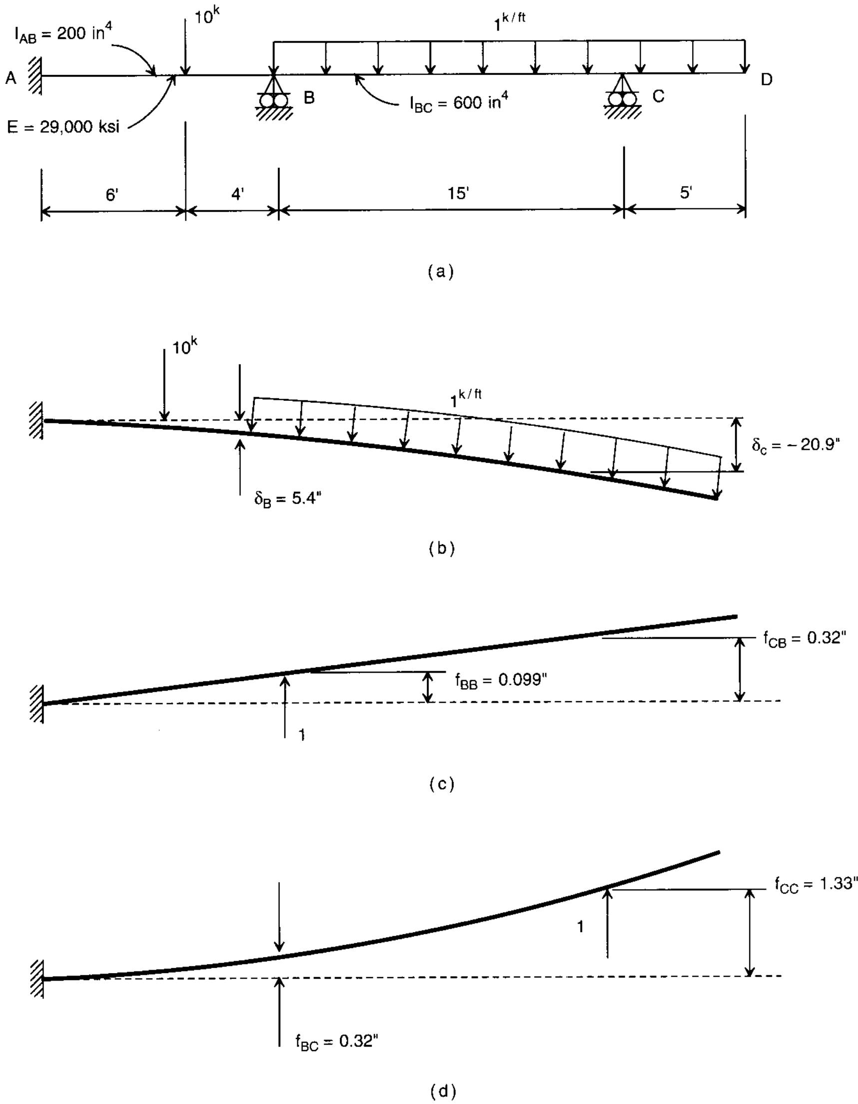


FIGURE 3.75 Analysis of a continuous beam by the force method. (*a*) Two-span beam with concentrated and uniform loads. (*b*) Displacements of beam when supports at *B* and *C* are removed. (*c*) Displacements for unit load at *B*. (*d*) Displacements for unit load at *C*.

$$
\begin{aligned}
P_1 &= k_{11}\Delta_1 + K_{12}\Delta_2 + \cdots + k_{1n}\Delta_n \\
P_2 &= k_{21}\Delta_1 + k_{22}\Delta_2 + \cdots + k_{2n}\Delta_n \\
&\vdots \\
P_n &= k_{n1}\Delta_1 + k_{n2}\Delta_2 + \cdots + k_{nn}\Delta_n
\end{aligned}
$$

or more compactly as

$$P_i = \sum_{j=1}^{n} k_{ij}\Delta_j \qquad \text{for } i = 1, 2, 3, \ldots, n \tag{3.134}$$

Member loads acting between degrees of freedom are converted to equivalent loads acting at these degrees of freedom.

The typical k_{ij} coefficient in Eq. (3.134) is a **stiffness coefficient.** It represents the resulting force (or moment) at point i in the direction of load P_i when a unit displacement at point j in the direction of Δ_j is imposed and all other degrees of freedom are restrained against displacement. P_i is the given concentrated load at degree of freedom i in the direction of Δ_i.

When loads, such as distributed loads, act between nodes, an equivalent force and moment should be determined for these nodes. For example, the nodal forces for one span of a continuous beam are the fixed-end moments and simple-beam reactions, both with signs reversed. **Fixed-end moments** for several beams under various loads are provided in Fig. 3.76. (See also Arts. 3.37, 3.38, and 3.39.)

(F. Arbabi, *Structural Analysis and Behavior,* McGraw-Hill, Inc., New York.)

3.37 SLOPE-DEFLECTION METHOD

One of several displacement methods for analyzing statically indeterminate structures that resist loads by bending involves use of slope-deflection equations. This method is convenient for analysis of continuous beams and rigid frames in which axial force effects may be neglected. It is not intended for analysis of trusses.

Consider a beam *AB* (Fig. 3.77*a*) that is part of a continuous structure. Under loading, the beam develops end moments M_{AB} at *A* and M_{BA} at *B* and end rotations θ_A and θ_B. The latter are the angles that the tangents to the deformed neutral axis at ends *A* and *B*, respectively; make with the original direction of the axis. (Counterclockwise rotations and moments are assumed positive.) Also, let Δ_{BA} be the displacement of *B* relative to *A* (Fig. 3.77*b*). For small deflections, the rotation of the chord joining *A* and *B* may be approximated by $\phi_{BA} = \Delta_{BA}/L$. The end moments, end rotations, and relative deflection are related by the slope-deflection equations:

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\phi_{BA}) + M_{AB}{}^F \tag{3.135a}$$

$$M_{BA} = \frac{2EI}{L}(\theta_A + 2\theta_B - 3\phi_{BA}) + M_{BA}{}^F \tag{3.135b}$$

where E = modulus of elasticity of the material
I = moment of inertia of the beam
L = span
$M_{AB}{}^F$ = fixed-end moment at *A*
$M_{BA}{}^F$ = fixed-end moment at *B*

Use of these equations for each member in a structure plus equations for equilibrium at the member connections is adequate for determination of member displacements. These displacements can then be substituted into the equations to determine the end moments.

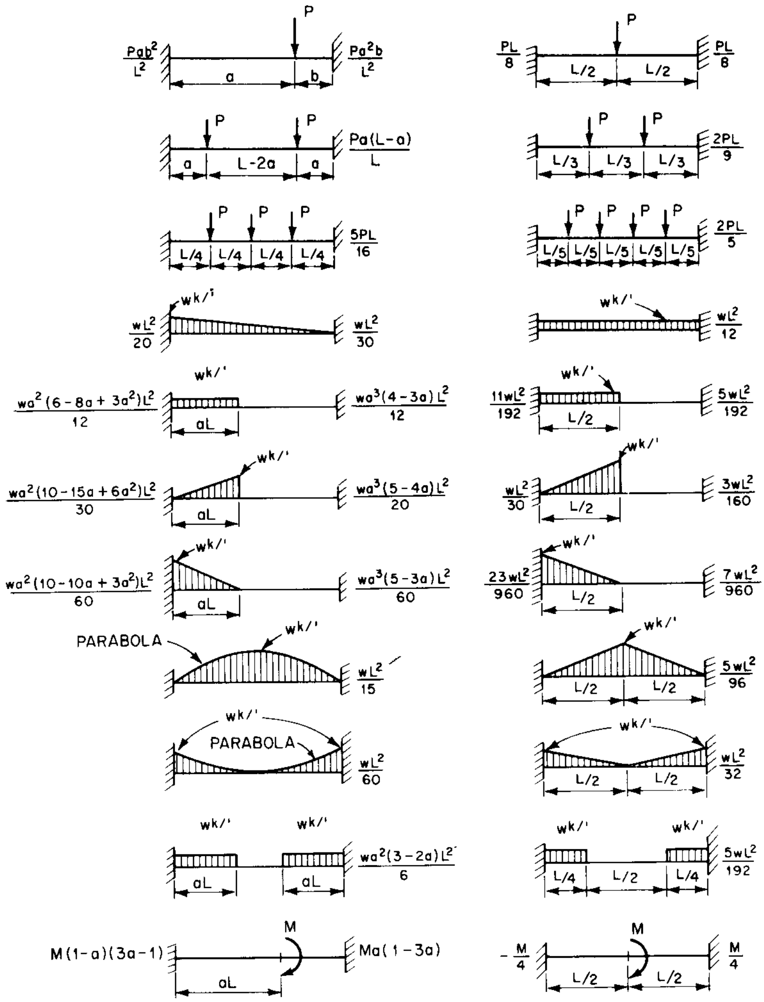


FIGURE 3.76 Fixed-end moments in beams.

As an example, the beam in Fig. 3.75*a* will be analyzed by employing the slope-deflection equations [Eqs. (3.135*a* and *b*)]. From Fig. 3.76, the fixed-end moments in span *AB* are

$$M_{AB}{}^{F} = \frac{10 \times 6 \times 4^2}{10^2} = 9.60 \text{ ft-kips}$$

$$M_{BA}{}^{F} = -\frac{10 \times 4 \times 6^2}{10^2} = -14.40 \text{ ft-kips}$$

The fixed-end moments in *BC* are

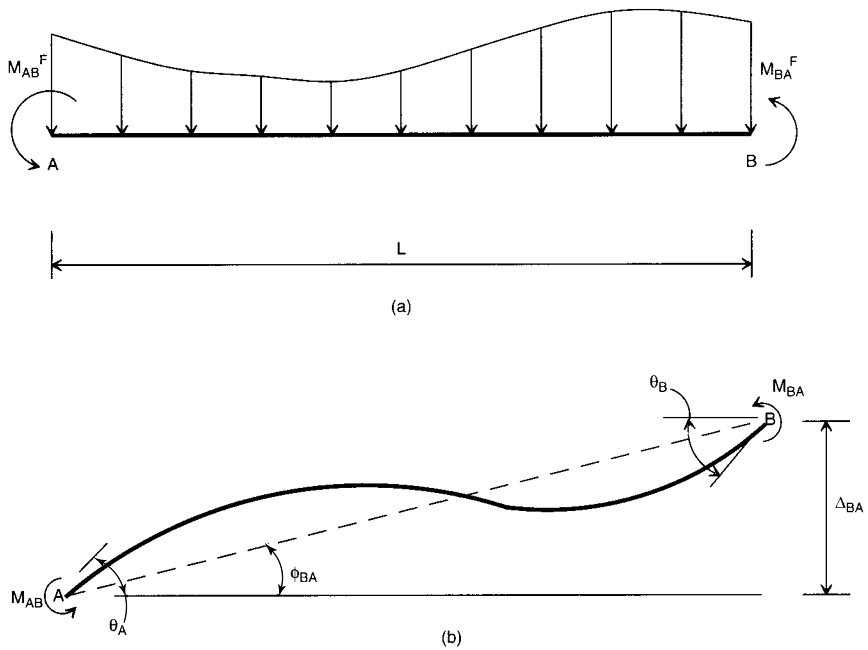


FIGURE 3.77 (*a*) Member of a continuous beam. (*b*) Elastic curve of the member for end moment and displacement of an end.

$$M_{BC}{}^F = 1 \times \frac{15^2}{12} = 18.75 \text{ ft-kips}$$

$$M_{CB}{}^F = -1 \times \frac{15^2}{12} = -18.75 \text{ ft-kips}$$

The moment at C from the cantilever is $M_{CD} = 12.50$ ft-kips.

If $E = 29{,}000$ ksi, $I_{AB} = 200$ in^4, and $I_{BC} = 600$ in^4, then $2EI_{AB}/L_{AB} = 8055.6$ ft-kips and $2EI_{BC}/L_{BC} = 16{,}111.1$ ft-kips. With $\theta_A = 0$, $\phi_{BA} = 0$, and $\phi_{CB} = 0$, Eq. (3.135) yields

$$M_{AB} = 8{,}055.6\theta_B + 9.60 \tag{3.136}$$

$$M_{BA} = 2 \times 8{,}055.6\theta_B - 14.40 \tag{3.137}$$

$$M_{BC} = 2 \times 16{,}111.1\theta_B + 16{,}111.1\theta_C + 18.75 \tag{3.138}$$

$$M_{CB} = 16{,}111.1\theta_B + 2 \times 16{,}111.1\theta_C - 18.75 \tag{3.139}$$

Also, equilibrium of joints B and C requires that

$$M_{BA} = -M_{BC} \tag{3.140}$$

$$M_{CB} = -M_{CD} = -12.50 \tag{3.141}$$

Substitution of Eqs. (3.137) and (3.138) in Eq. (3.140) and Eq. (3.139) in Eq. (3.141) gives

$$48{,}333.4\theta_B + 16{,}111.1\theta_C = -4.35 \tag{3.142}$$

$$16{,}111.1\theta_B + 32{,}222.2\theta_C = 6.25 \tag{3.143}$$

Solution of these equations yields $\theta_B = -1.86 \times 10^{-4}$ and $\theta_C = 2.87 \times 10^{-4}$ radians. Substitution in Eqs. (3.136) to (3.139) gives the end moments: $M_{AB} = 8.1$, $M_{BA} = -17.4$, $M_{BC} = 17.4$, and $M_{CB} = -12.5$ ft-kips. With these moments and switching the signs of moments at the left end of members to be consistent with the sign convention in Art. 3.18, the shear and bending-moment diagrams shown in Fig. 3.78*a* and *b* can be obtained. This example also demonstrates that a valuable by-product of the displacement method is the calculation of several of the node displacements.

If axial force effects are neglected, the slope-deflection method also can be used to analyze rigid frames.

(J. McCormac and R. E. Elling, *Structural Analysis—A Classical and Matrix Approach,* Harper and Row Publishers, New York; and R. C. Hibbeler, *Structural Analysis,* Prentice Hall, New Jersey.)

3.38 MOMENT-DISTRIBUTION METHOD

The moment-distribution method is one of several displacement methods for analyzing continuous beams and rigid frames. Moment distribution, however, provides an alternative to solving the system of simultaneous equations that result with other methods, such as slope deflection. (See Arts. 3.36, 3.37, and 3.39.)

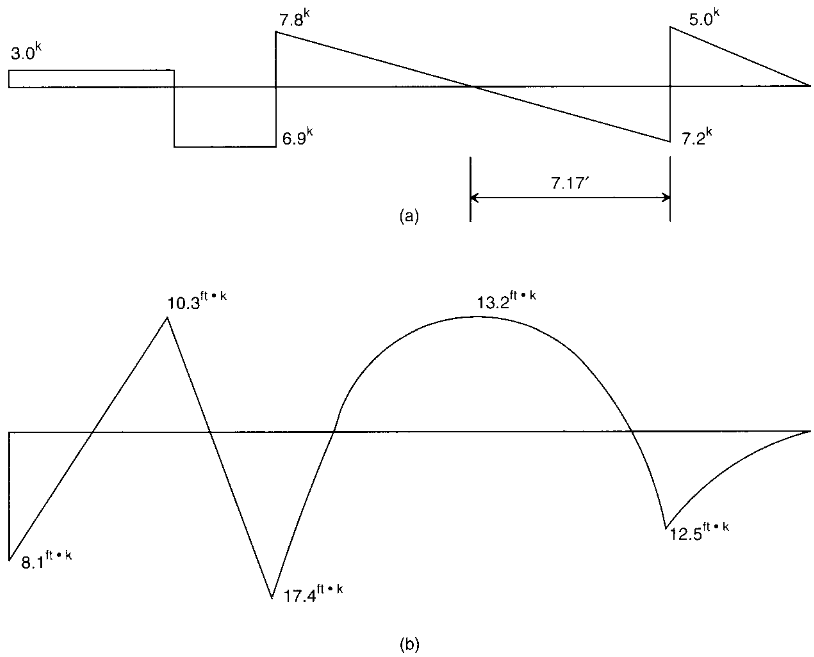


FIGURE 3.78 Shear diagram (*a*) and moment diagram (*b*) for the continuous beam in Fig. 3.75*a*.

Moment distribution is based on the fact that the bending moment at each end of a member of a continuous frame equals the sum of the fixed-end moments due to the applied loads on the span and the moments produced by rotation of the member ends and of the chord between these ends. Given fixed-end moments, the moment-distribution method determines moments generated when the structure deforms.

Figure 3.79 shows a structure consisting of three members rigidly connected at joint O (ends of the members at O must rotate the same amount). Supports at A, B, and C are fixed (rotation not permitted). If joint O is locked temporarily to prevent rotation, applying a load on member OA induces fixed-end moments at A and O. Suppose fixed-end moment $M_{OA}{}^F$ induces a counterclockwise moment on locked joint O. Now, if the joint is released, $M_{OA}{}^F$ rotates it counterclockwise. Bending moments are developed in each member joined at O to balance $M_{OA}{}^F$. Bending moments are also developed at the fixed supports A, B, and C. These moments are said to be carried over from the moments in the ends of the members at O when the joint is released.

The total end moment in each member at O is the algebraic sum of the fixed-end moment before release and the moment in the member at O caused by rotation of the joint, which depends on the relative stiffnesses of the members. Stiffness of a prismatic fixed-end beam is proportional to EI/L, where E is the modulus of elasticity, I the moment of inertia, and L the span.

When a fixed joint is unlocked, it rotates if the algebraic sum of the bending moments at the joint does not equal zero. The moment that causes the joint to rotate is the **unbalanced moment.** The moments developed at the far ends of each member of the released joint when the joint rotates are **carry-over moments.**

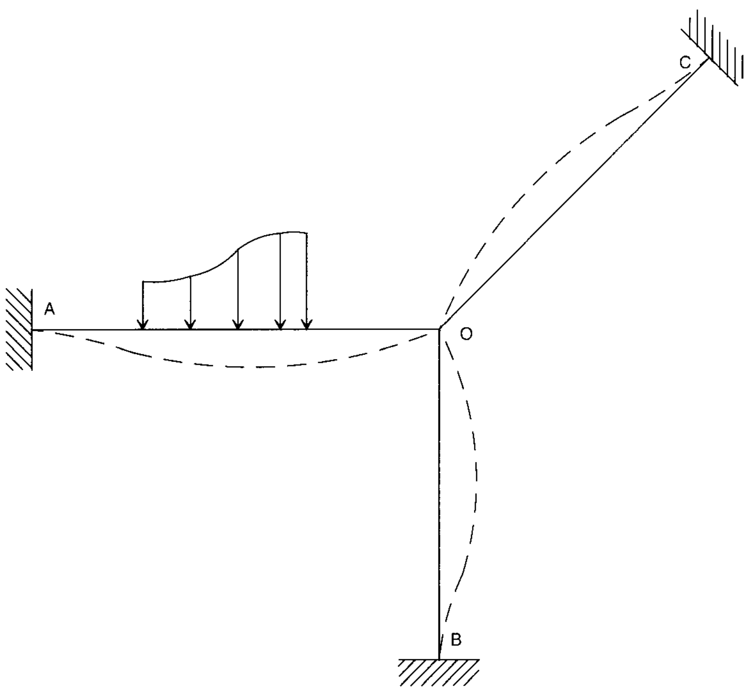


FIGURE 3.79 Straight members rigidly connected at joint O. Dash lines show deformed shape after loading.

In general, if all joints are locked and then one is released, the amount of unbalanced moment distributed to member i connected to the unlocked joint is determined by the **distribution factor** D_i the ratio of the moment distributed to i to the unbalanced moment. For a prismatic member,

$$D_i = \frac{E_i I_i / L_i}{\sum_{j=1}^{n} E_j I_j / L_j} \tag{3.144}$$

where $\Sigma_{j=1}^{n} E_j I_j / L_j$ is the sum of the stiffness of all n members, including member i, joined at the unlocked joint. Equation (3.144) indicates that the sum of all distribution factors at a joint should equal 1.0. Members cantilevered from a joint contribute no stiffness and therefore have a distribution factor of zero.

The amount of moment distributed from an unlocked end of a prismatic member to a locked end is ½. This **carry-over factor** can be derived from Eqs. (3.135*a* and *b*) with $\theta_A = 0$.

Moments distributed to fixed supports remain at the support; i.e., fixed supports are never unlocked. At a pinned joint (non-moment-resisting support), all the unbalanced moment should be distributed to the pinned end on unlocking the joint. In this case, the distribution factor is 1.0.

To illustrate the method, member end moments will be calculated for the continuous beam shown in Fig. 3.75*a*. All joints are initially locked. The concentrated load on span *AB* induces fixed-end moments of 9.60 and −14.40 ft-kips at *A* and *B*, respectively (see Art. 3.37). The uniform load on *BC* induces fixed-end moments of 18.75 and −18.75 ft-kips at *B* and *C*, respectively. The moment at *C* from the cantilever *CD* is 12.50 ft-kips. These values are shown in Fig. 3.80*a*.

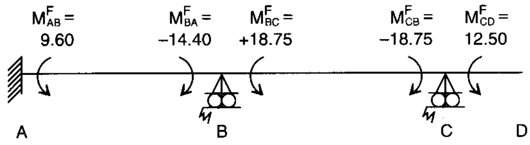
The distribution factors at joints where two or more members are connected are then calculated from Eq. (3.144). With $EI_{AB}/L_{AB} = 200E/120 = 1.67E$ and $EI_{BC}/L_{BC} = 600E/180 = 3.33E$, the distribution factors are $D_{BA} = 1.67E/(1.67E + 3.33E) = 0.33$ and $D_{BC} = 3.33/5.00 = 0.67$. With $EI_{CD}/L_{CD} = 0$ for a cantilevered member, $D_{CB} = 10E/(0 + 10E) = 1.00$ and $D_{CD} = 0.00$.

Joints not at fixed supports are then unlocked one by one. In each case, the unbalanced moments are calculated and distributed to the ends of the members at the unlocked joint according to their distribution factors. The distributed end moments, in turn, are "carried over" to the other end of each member by multiplication of the distributed moment by a carry-over factor of ½. For example, initially unlocking joint *B* results in an unbalanced moment of $-14.40 + 18.75 = 4.35$ ft-kips. To balance this moment, −4.35 ft-kips is distributed to members *BA* and *BC* according to their distribution factors: $M_{BA} = -4.35D_{BA} = -4.35 \times 0.33 = -1.44$ ft-kips and $M_{BC} = -4.35D_{BC} = -2.91$ ft-kips. The carry-over moments are $M_{AB} = M_{BA}/2 = -0.72$ and $M_{CB} = M_{BC}/2 = -1.46$. Joint *B* is then locked, and the resulting moments at each member end are summed: $M_{AB} = 9.60 - 0.72 = 8.88$, $M_{BA} = -14.40 - 1.44 = -15.84$, $M_{BC} = 18.75 - 2.91 = 15.84$, and $M_{CB} = -18.75 - 1.46 = -20.21$ ft-kips. When the step is complete, the moments at the unlocked joint balance, that is, $-M_{BA} = M_{BC}$.

The procedure is then continued by unlocking joint *C*. After distribution of the unbalanced moments at *C* and calculation of the carry-over moment to *B*, the joint is locked, and the process is repeated for joint *B*. As indicated in Fig. 3.80*b*, iterations continue until the final end moments at each joint are calculated to within the designer's required tolerance.

There are several variations of the moment-distribution method. This method may be extended to determine moments in rigid frames that are subject to drift, or sidesway.

(C. H. Norris et al., *Elementary Structural Analysis,* 4th ed., McGraw-Hill, Inc., New York; J. McCormac and R. E. Elling, *Structural Analysis—A Classical and Matrix Approach,* Harper and Row Publishers, New York.)



(a)

		D_{BA}	D_{BC}		D_{CB}	D_{CD}	
		0.33	0.67		1.00	0.00	
+9.60		−14.40	+18.75		−18.75	+12.50	joints all locked
−0.72	←	−1.44	−2.91	→	−1.46	—	
+8.88		−15.84	+15.84		−20.21	+12.50	after unlocking B
—		—	+3.86	←	+7.71	0.00	
+8.88		−15.84	+19.70		−12.50	+12.50	after unlocking C
−0.64	←	−1.27	−2.59	→	−1.30	—	
+8.24		−17.11	+17.11		−13.80	+12.50	after unlocking B
—		—	+0.65	←	+1.30	0.00	
+8.24		−17.11	17.76		−12.50	+12.50	after unlocking C
−0.11	←	−0.21	−0.44	→	−0.22	—	
+8.13		−17.32	+17.32		−12.72	+12.50	after unlocking B
—		—	+0.11	←	+0.22	0.00	
+8.13		−17.32	17.44		−12.50	+12.50	after unlocking C
−0.02	←	−0.04	−0.08	→	−0.04	—	
+8.11		−17.36	+17.36		−12.54	+12.50	after unlocking B
8.1		−17.4	+17.4		−12.5	(+12.5)	

(b)

FIGURE 3.80 (*a*) Fixed-end moments for beam in Fig. 3.75*a*. (*b*) Steps in moment distribution. Fixed-end moments are given in the top line, final moments in the bottom line, in ft-kips.

3.39 MATRIX STIFFNESS METHOD

As indicated in Art. 3.36, displacement methods for analyzing structures relate force components acting at the joints, or **nodes,** to the corresponding displacement components at these joints through a set of equilibrium equations. In matrix notation, this set of equations [Eq. (3.134)] is represented by

$$\mathbf{P} = \mathbf{K}\boldsymbol{\Delta} \tag{3.145}$$

where $\mathbf{P}$ = column vector of nodal external load components $\{P_1, P_2, \ldots, P_n\}^T$
$\mathbf{K}$ = stiffness matrix for the structure
$\boldsymbol{\Delta}$ = column vector of nodal displacement components: $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}^T$
n = total number of degrees of freedom
T = transpose of a matrix (columns and rows interchanged)

A typical element k_{ij} of $\mathbf{K}$ gives the load at nodal component i in the direction of load component P_i that produces a unit displacement at nodal component j in the direction of displacement component Δ_j. Based on the reciprocal theorem (see Art. 3.25), the square matrix K is symmetrical, that is, $k_{ij} = k_{ji}$.

For a specific structure, Eq. (3.145) is generated by first writing equations of equilibrium at each node. Each force and moment component at a specific node must be balanced by the sum of member forces acting at that joint. For a two-dimensional frame defined in the

xy plane, force and moment components per node include F_x, F_y, and M_z. In a three-dimensional frame, there are six force and moment components per node: F_x, F_y, F_z, M_x, M_y, and M_z.

From member force-displacement relationships similar to Eq. (3.135), member force components in the equations of equilibrium are replaced with equivalent displacement relationships. The resulting system of equilibrium equations can be put in the form of Eq. (3.145).

Nodal boundary conditions are then incorporated into Eq. (3.145). If, for example, there are a total of n degrees of freedom, of which m degrees of freedom are restrained from displacement, there would be $n - m$ unknown displacement components and m unknown restrained force components or reactions. Hence a total of $(n - m) + m = n$ unknown displacements and reactions could be determined by the system of n equations provided with Eq. (3.145).

Once all displacement components are known, member forces may be determined from the member force-displacement relationships.

For a prismatic member subjected to the end forces and moments shown in Fig. 3.81a, displacements at the ends of the member are related to these member forces by the matrix expression

$$\begin{Bmatrix} F'_{xi} \\ F'_{yi} \\ M'_{zi} \\ F'_{xj} \\ F'_{yj} \\ M'_{zj} \end{Bmatrix} = \frac{E}{L^3} \begin{bmatrix} AL^2 & 0 & 0 & -AL^2 & 0 & 0 \\ 0 & 12I & 6IL & 0 & -12I & 6IL \\ 0 & 6IL & 4IL^2 & 0 & -6IL & 2IL^2 \\ -AL^2 & 0 & 0 & AL^2 & 0 & 0 \\ 0 & -12I & -6IL & 0 & 12I & -6IL \\ 0 & 6IL & 2IL^2 & 0 & -6IL & 4IL^2 \end{bmatrix} \begin{Bmatrix} \Delta'_{xi} \\ \Delta'_{yi} \\ \theta'_{zi} \\ \Delta'_{xj} \\ \Delta'_{yj} \\ \theta'_{zj} \end{Bmatrix} \tag{3.146}$$

where L = length of member (distance between i and j)
E = modulus of elasticity
A = cross-sectional area of member
I = moment of inertia about neutral axis in bending

In matrix notation, Eq. (3.146) for the ith member of a structure can be written

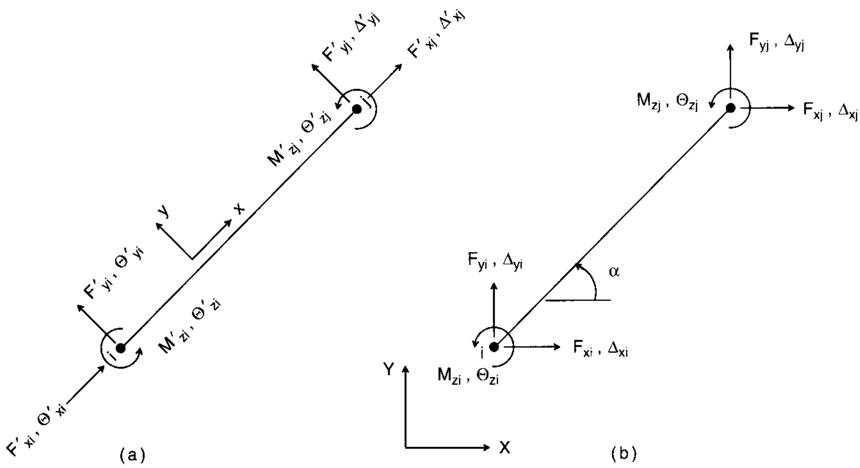


FIGURE 3.81 Member of a continuous structure. (a) Forces at the ends of the member and deformations are given with respect to the member local coordinate system; (b) with respect to the structure global coordinate system.

$$\mathbf{S}_i' = \mathbf{k}_i'\boldsymbol{\delta}_i' \tag{3.147}$$

where $\mathbf{S}_i'$ = vector forces and moments acting at the ends of member i
$\mathbf{k}_i'$ = stiffness matrix for member i
$\boldsymbol{\delta}_i'$ = vector of deformations at the ends of member i

The force-displacement relationships provided by Eqs. (3.146) and (3.147) are based on the member's xy **local coordinate system** (Fig. 3.81*a*). If this coordinate system is not aligned with the structure's XY **global coordinate system,** these equations must be modified or transformed. After transformation of Eq. (3.147) to the global coordinate system, it would be given by

$$\mathbf{S}_i = \mathbf{k}_i\boldsymbol{\delta}_i \tag{3.148}$$

where $\mathbf{S}_i = \boldsymbol{\Gamma}_i^T\mathbf{S}_i'$ = force vector for member i, referenced to global coordinates
$\mathbf{k}_i = \boldsymbol{\Gamma}_i^T\mathbf{k}_i'\boldsymbol{\Gamma}_i$ = member stiffness matrix
$\boldsymbol{\delta}_i = \boldsymbol{\Gamma}_i^T\boldsymbol{\delta}_i'$ = displacement vector for member i, referenced to global coordinates
$\boldsymbol{\Gamma}_i$ = transformation matrix for member i

For the member shown in Fig. 3.81*b*, which is defined in two-dimensional space, the **transformation matrix** is

$$\boldsymbol{\Gamma} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.149}$$

where α = angle measured from the structure's global X axis to the member's local x axis.

Example. To demonstrate the matrix displacement method, the rigid frame shown in Fig. 3.82*a*. will be analyzed. The two-dimensional frame has three joints, or nodes, A, B, and C, and hence a total of nine possible degrees of freedom (Fig. 3.82*b*). The displacements at node A are not restrained. Nodes B and C have zero displacement. For both AB and AC, modulus of elasticity $E = 29{,}000$ ksi, area $A = 1$ in^2, and moment of inertia $I = 10$ in^4. Forces will be computed in kips; moments, in kip-in.

At each degree of freedom, the external forces must be balanced by the member forces. This requirement provides the following equations of equilibrium with reference to the global coordinate system:

At the free degree of freedom at node A, $\Sigma F_{xA} = 0$, $\Sigma F_{yA} = 0$, and $\Sigma M_{zA} = 0$:

$$10 = F_{xAB} + F_{xAC} \tag{3.150a}$$

$$-200 = F_{yAB} + F_{yAC} \tag{3.150b}$$

$$0 = M_{zAB} + M_{zAC} \tag{3.150c}$$

At the restrained degrees of freedom at node B, $\Sigma F_{xB} = 0$, $\Sigma F_{yB} = 0$, and $\Sigma M_{zA} = 0$:

$$R_{xB} - F_{xBA} = 0 \tag{3.151a}$$

$$R_{yB} - F_{yBA} = 0 \tag{3.151b}$$

$$M_{zB} - M_{zBA} = 0 \tag{3.151c}$$

At the restrained degrees of freedom at node C, $\Sigma F_{xC} = 0$, $\Sigma F_{yC} = 0$, and $\Sigma M_{zC} = 0$:

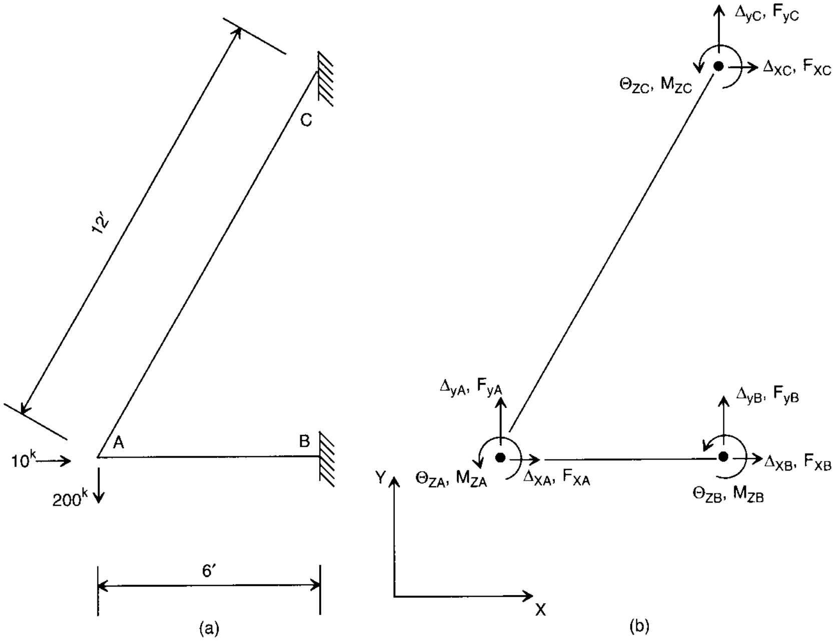


FIGURE 3.82 (*a*) Two-member rigid frame, with modulus of elasticity $E = 29{,}000$ ksi, area $A = 1$ in^2, and moment of inertia $I = 10$ in^4. (*b*) Degrees of freedom at nodes.

$$R_{xC} - F_{xCA} = 0 \tag{3.152a}$$

$$R_{yC} - F_{yCA} = 0 \tag{3.152b}$$

$$M_{zC} - M_{zCA} = 0 \tag{3.152c}$$

where subscripts identify the direction, member, and degree of freedom.

Member force components in these equations are then replaced by equivalent displacement relationships with the use of Eq. (3.148). With reference to the global coordinates, these relationships are as follows:

For member AB with $\alpha = 0°$, $\mathbf{S}_{AB} = \mathbf{\Gamma}_{AB}^T \mathbf{k}'_{AB} \mathbf{\Gamma}_{AB} \boldsymbol{\delta}_{AB}$:

$$\begin{Bmatrix} F_{xAB} \\ F_{yAB} \\ M_{zAB} \\ F_{xBA} \\ F_{yBA} \\ M_{zBA} \end{Bmatrix} = \begin{bmatrix} 402.8 & 0 & 0 & -402.8 & 0 & 0 \\ 0 & 9.324 & 335.6 & 0 & -9.324 & 335.6 \\ 0 & 335.6 & 16111 & 0 & -335.6 & 8056 \\ -402.8 & 0 & 0 & 402.8 & 0 & 0 \\ 0 & -9.324 & -335.6 & 0 & 9.324 & -335.6 \\ 0 & 335.6 & 8056 & 0 & -335.6 & 16111 \end{bmatrix} \begin{Bmatrix} \Delta_{xA} \\ \Delta_{yA} \\ \Theta_{zA} \\ \Delta_{xB} \\ \Delta_{yB} \\ \Theta_{zB} \end{Bmatrix} \tag{3.153}$$

For member AC with $\alpha = 60°$, $\mathbf{S}_{AC} = \mathbf{\Gamma}_{AC}^T \mathbf{k}'_{AC} \mathbf{\Gamma}_{AC} \boldsymbol{\delta}_{AC}$:

$$\begin{Bmatrix} F_{xAC} \\ F_{yAC} \\ M_{zAC} \\ F_{xCA} \\ F_{yCA} \\ M_{zCA} \end{Bmatrix} = \begin{bmatrix} 51.22 & 86.70 & -72.67 & -51.22 & -86.70 & -72.67 \\ 86.70 & 151.3 & 41.96 & -86.70 & -151.3 & 41.96 \\ -72.67 & 41.96 & 8056 & 72.67 & -41.96 & 4028 \\ -51.22 & -86.70 & 72.67 & 51.22 & 86.70 & 72.67 \\ -86.70 & -151.3 & -41.96 & 86.70 & 151.3 & -41.96 \\ -72.67 & 41.96 & 4028 & 72.67 & -41.96 & 8056 \end{bmatrix} \begin{Bmatrix} \Delta_{xA} \\ \Delta_{yA} \\ \Theta_{zA} \\ \Delta_{xC} \\ \Delta_{yC} \\ \Theta_{zC} \end{Bmatrix} \tag{3.154}$$

Incorporating the support conditions $\Delta_{xB} = \Delta_{yB} = \Theta_{zB} = \Delta_{xC} = \Delta_{yC} = \Theta_{zC} = 0$ into Eqs. (3.153) and (3.154) and then substituting the resulting displacement relationships for the member forces in Eqs. (3.150) to (3.152) yields

$$\begin{Bmatrix} 10 \\ -200 \\ 0 \\ R_{xB} \\ R_{yB} \\ M_{zB} \\ R_{xC} \\ R_{yC} \\ M_{zC} \end{Bmatrix} = \begin{bmatrix} 402.8 + 51.22 & 0 + 86.70 & 0 - 72.67 \\ 0 + 86.70 & 9.324 + 151.3 & 335.6 + 41.96 \\ 0 - 72.67 & 335.6 + 41.96 & 16111 + 8056 \\ -402.8 & 0 & 0 \\ 0 & -9.324 & -335.6 \\ 0 & 335.6 & 8056 \\ -51.22 & -86.70 & 72.67 \\ -86.70 & -151.3 & -41.96 \\ -72.67 & 41.96 & 4028 \end{bmatrix} \begin{Bmatrix} \Delta_{xA} \\ \Delta_{yA} \\ \Theta_{zA} \end{Bmatrix} \tag{3.155}$$

Equation (3.155) contains nine equations with nine unknowns. The first three equations may be used to solve the displacements at the free degrees of freedom $\mathbf{\Delta}_f = \mathbf{K}_{ff}^{-1}\mathbf{P}_f$:

$$\begin{Bmatrix} \Delta_{xA} \\ \Delta_{yA} \\ \Theta_{zA} \end{Bmatrix} = \begin{bmatrix} 454.0 & 86.70 & -72.67 \\ 86.70 & 160.6 & 377.6 \\ -72.67 & 377.6 & 24167 \end{bmatrix}^{-1} \begin{Bmatrix} 10 \\ -200 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.3058 \\ -1.466 \\ 0.0238 \end{Bmatrix} \tag{3.156a}$$

These displacements may then be incorporated into the bottom six equations of Eq. (3.155) to solve for the unknown reactions at the restrained nodes, $\mathbf{P}_s = \mathbf{K}_{sf}\mathbf{\Delta}_f$:

$$\begin{Bmatrix} R_{xB} \\ R_{yB} \\ M_{zB} \\ R_{xC} \\ R_{yC} \\ M_{zC} \end{Bmatrix} = \begin{bmatrix} -402.8 & 0 & 0 \\ 0 & -9.324 & -335.6 \\ 0 & 335.6 & 8056 \\ -51.22 & -86.70 & 72.67 \\ -86.70 & -151.3 & -41.96 \\ -72.67 & 41.96 & 4028 \end{bmatrix} \begin{Bmatrix} 0.3058 \\ -1.466 \\ 0.0238 \end{Bmatrix} = \begin{Bmatrix} -123.2 \\ 5.67 \\ -300.1 \\ 113.2 \\ 194.3 \\ 12.2 \end{Bmatrix} \tag{3.156b}$$

With all displacement components now known, member end forces may be calculated. Displacement components that correspond to the ends of a member should be transformed from the global coordinate system to the member's local coordinate system, $\boldsymbol{\delta}' = \boldsymbol{\Gamma}\boldsymbol{\delta}$.

For member *AB* with $\alpha = 0°$:

$$\begin{Bmatrix} \Delta'_{xA} \\ \Delta'_{yA} \\ \Theta'_{zA} \\ \Delta'_{xB} \\ \Delta'_{yB} \\ \Theta'_{zB} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.3058 \\ -1.466 \\ 0.0238 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.3058 \\ -1.466 \\ 0.0238 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.157a}$$

For member *AC* with $\alpha = 60°$:

$$\begin{Bmatrix} \Delta'_{xA} \\ \Delta'_{yA} \\ \Theta'_{zA} \\ \Delta'_{xC} \\ \Delta'_{yC} \\ \Theta'_{zC} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.866 & 0 & 0 & 0 & 0 \\ -0.866 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & -0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.3058 \\ -1.466 \\ 0.0238 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.1117 \\ -0.9978 \\ 0.0238 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.157b}$$

Member end forces are then obtained by multiplying the member stiffness matrix by the membr end displacements, both with reference to the member local coordinate system, $\mathbf{S}' = \mathbf{k}'\boldsymbol{\delta}'$.

For member *AB* in the local coordinate system:

$$\begin{Bmatrix} F'_{xAB} \\ F'_{yAB} \\ M'_{zAB} \\ F'_{xBA} \\ F'_{yBA} \\ M'_{zBA} \end{Bmatrix} = \begin{bmatrix} 402.8 & 0 & 0 & -402.8 & 0 & 0 \\ 0 & 9.324 & 335.6 & 0 & -9.324 & 335.6 \\ 0 & 335.6 & 16111 & 0 & -335.6 & 8056 \\ -402.8 & 0 & 0 & 402.8 & 0 & 0 \\ 0 & -9.324 & -335.6 & 0 & 9.324 & -335.6 \\ 0 & 335.6 & 8056 & 0 & -335.6 & 16111 \end{bmatrix} \times \begin{Bmatrix} 0.3058 \\ -1.466 \\ 0.0238 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 123.2 \\ -5.67 \\ -108.2 \\ -123.2 \\ 5.67 \\ -300.1 \end{Bmatrix} \tag{3.158}$$

For member *AC* in the local coordinate system:

$$\begin{Bmatrix} F'_{xAC} \\ F'_{yAC} \\ M'_{zAC} \\ F'_{xCA} \\ F'_{yCA} \\ M'_{zCA} \end{Bmatrix} = \begin{bmatrix} 201.4 & 0 & 0 & -201.4 & 0 & 0 \\ 0 & 1.165 & 83.91 & 0 & -1.165 & 83.91 \\ 0 & 83.91 & 8056 & 0 & -83.91 & 4028 \\ -201.4 & 0 & 0 & 201.4 & 0 & 0 \\ 0 & -1.165 & -83.91 & 0 & 1.165 & -83.91 \\ 0 & 83.91 & 4028 & 0 & -83.91 & 8056 \end{bmatrix} \times \begin{Bmatrix} -1.1117 \\ -0.9978 \\ 0.0238 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -224.9 \\ 0.836 \\ 108.2 \\ 224.9 \\ -0.836 \\ 12.2 \end{Bmatrix} \tag{3.159}$$

At this point all displacements, member forces, and reaction components have been determined.

The matrix displacement method can be used to analyze both determinate and indeterminate frames, trusses, and beams. Because the method is based primarily on manipulating matrices, it is employed in most structural-analysis computer programs. In the same context, these programs can handle substantial amounts of data, which enables analysis of large and often complex structures.

(W. McGuire, R. H. Gallagher and R. D. Ziemian, *Matrix Structural Analysis,* John Wiley & Sons Inc., New York; D. L. Logan, *A First Course in the Finite Element Method,* PWS-Kent Publishing, Boston, Mass.)

3.40 INFLUENCE LINES

In studies of the variation of the effects of a moving load, such as a reaction, shear, bending moment, or stress, at a given point in a structure, use of diagrams called **influence lines** is helpful. An influence line is a diagram showing the variation of an effect as a unit load moves over a structure.

An influence line is constructed by plotting the position of the unit load as the abscissa and as the ordinate at that position, to some scale, the value of the effect being studied. For example, Fig. 3.83*a* shows the influence line for reaction *A* in simple-beam *AB*. The sloping line indicates that when the unit load is at *A*, the reaction at *A* is 1.0. When the load is at *B*, the reaction at *A* is zero. When the unit load is at midspan, the reaction at *A* is 0.5. In general, when the load moves from *B* toward *A*, the reaction at *A* increases linearly: $R_A = (L - x)/L$, where x is the distance from *A* to the position of the unit load.

Figure 3.83*b* shows the influence line for shear at the quarter point *C*. The sloping lines indicate that when the unit load is at support *A* or *B*, the shear at *C* is zero. When the unit load is a small distance to the left of *C*, the shear at *C* is −0.25; when the unit load is a small distance to the right of *C*, the shear at *C* is 0.75. The influence line for shear is linear on each side of *C*.

Figure 3.83*c* and *d* show the influence lines for bending moment at midspan and quarter point, respectively. Figures 3.84 and 3.85 give influence lines for a cantilever and a simple beam with an overhang.

Influence lines can be used to calculate reactions, shears, bending moments, and other effects due to fixed and moving loads. For example, Fig. 3.86*a* shows a simply supported beam of 60-ft span subjected to a dead load $w = 1.0$ kip per ft and a live load consisting of three concentrated loads. The reaction at *A* due to the dead load equals the product of the area under the influence line for the reaction at *A* (Fig. 3.86*b*) and the uniform load *w*. The maximum reaction at *A* due to the live loads may be obtained by placing the concentrated loads as shown in Fig. 3.86*b* and equals the sum of the products of each concentrated load and the ordinate of the influence line at the location of the load. The sum of the dead-load reaction and the maximum live-load reaction therefore is

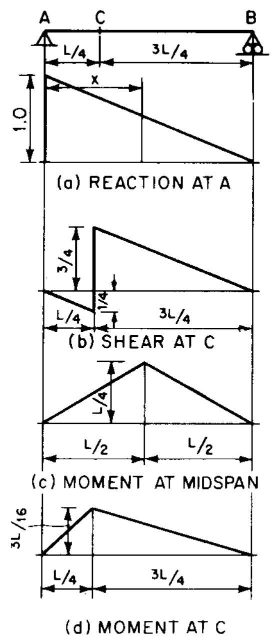


FIGURE 3.83 Influence diagrams for a simple beam.

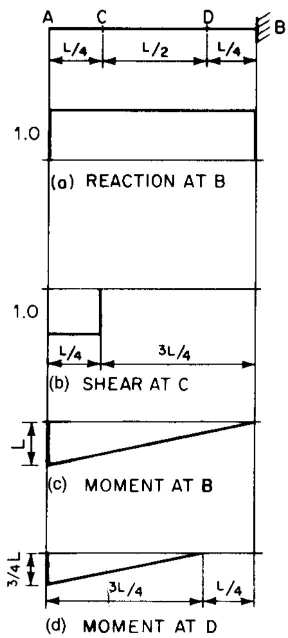


FIGURE 3.84 Influence diagrams for a cantilever.

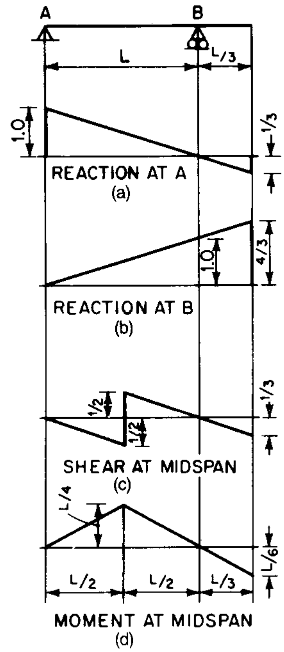


FIGURE 3.85 Influence diagrams for a beam with overhang.

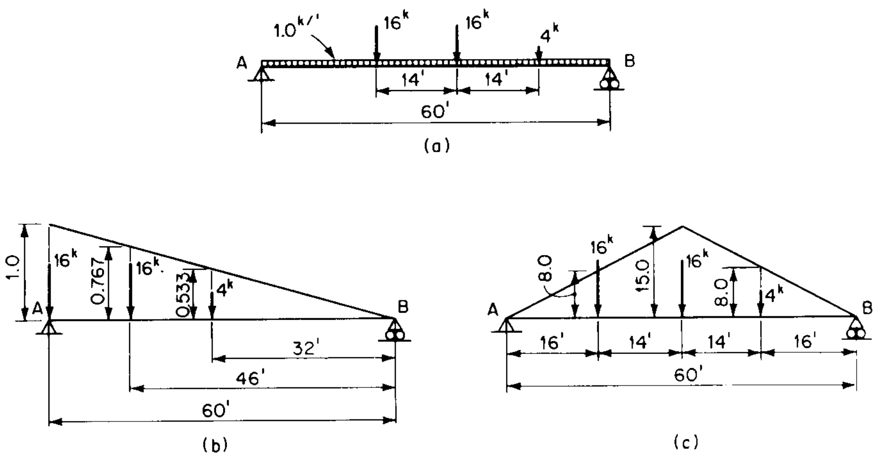


FIGURE 3.86 Determination for moving loads on a simple beam (*a*) of maximum end reaction (*b*) and maximum midspan moment (*c*) from influence diagrams.

$$R_A = \tfrac{1}{2} \times 1.0 \times 60 \times 1.0 + 16 \times 1.0 + 16 \times 0.767 + 4 \times 0.533 = 60.4 \text{ kips}$$

Figure 3.86*c* is the influence diagram for midspan bending moment with a maximum ordinate $L/4 = 60/4 = 15$. Figure 3.86*c* also shows the influence diagram with the live loads positioned for maximum moment at midspan. The dead load moment at midspan is the product of w and the area under the influence line. The midspan live-load moment equals the sum of the products of each live load and the ordinate at the location of each load. The sum of the dead-load moment and the maximum live-load moment equals

$$M = \tfrac{1}{2} \times 15 \times 60 \times 1.0 + 16 \times 15 + 16 \times 8 + 4 \times 8 = 850 \text{ ft-kips}$$

An important consequence of the reciprocal theorem presented in Art. 3.25 is the **Mueller-Breslau principle:** The influence line of a certain effect is to some scale the deflected shape of the structure when that effect acts.

The effect, for example, may be a reaction, shear, moment, or deflection at a point. This principle is used extensively in obtaining influence lines for statically indeterminate structures (see Art. 3.28).

Figure 3.87*a* shows the influence line for reaction at support B for a two-span continuous beam. To obtain this influence line, the support at B is replaced by a unit upward-concentrated load. The deflected shape of the beam is the influence line of the reaction at point B to some

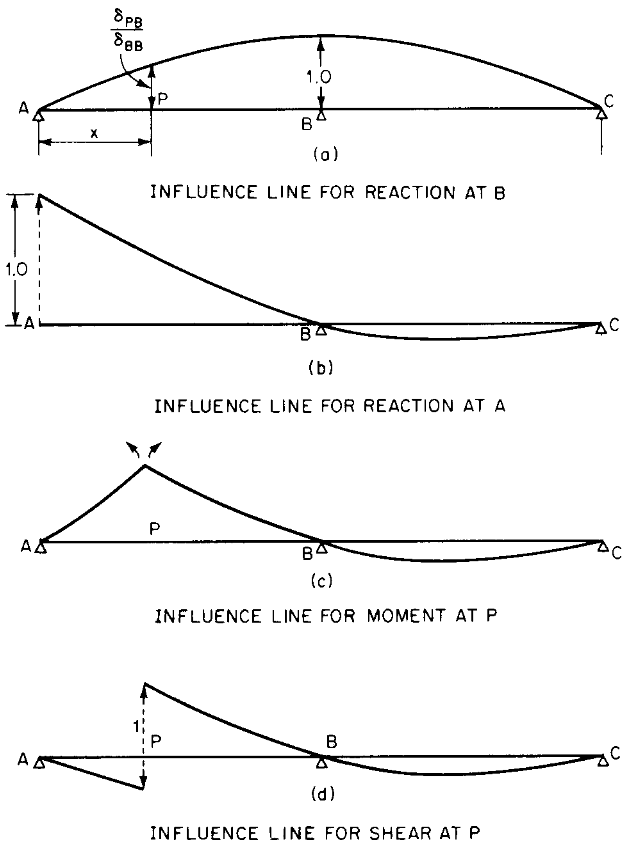


FIGURE 3.87 Influence lines for a two-span continuous beam.

scale. To show this, let δ_{BP} be the deflection at B due to a unit load at any point P when the support at B is removed, and let δ_{BB} be the deflection at B due to a unit load at B. Since, actually, reaction R_B prevents deflection at B, $R_B\delta_{BB} - \delta_{BP} = 0$. Thus $R_B = \delta_{BP}/\delta_{BB}$. By Eq. (3.124), however, $\delta_{BP} = \delta_{PB}$. Hence

$$R_B = \frac{\delta_{BP}}{\delta_{BB}} = \frac{\delta_{PB}}{\delta_{BB}} \tag{3.160}$$

Since δ_{BB} is constant, R_B is proportional to δ_{PB}, which depends on the position of the unit load. Hence the influence line for a reaction can be obtained from the deflection curve resulting from replacement of the support by a unit load. The magnitude of the reaction may be obtained by dividing each ordinate of the deflection curve by the displacement of the support due to a unit load applied there.

Similarly, influence lines may be obtained for reaction at A and moment and shear at P by the Mueller-Breslau principle, as shown in Figs. 3.87*b*, *c*, and *d*, respectively.

(C. H. Norris et al., *Elementary Structural Analysis;* and F. Arbabi, *Structural Analysis and Behavior,* McGraw-Hill, Inc., New York.)

INSTABILITY OF STRUCTURAL COMPONENTS

3.41 ELASTIC FLEXURAL BUCKLING OF COLUMNS

A member subjected to pure compression, such as a column, can fail under axial load in either of two modes. One is characterized by excessive axial deformation and the second by **flexural buckling** or excessive lateral deformation.

For short, stocky columns, Eq. (3.48) relates the axial load P to the compressive stress f. After the stress exceeds the yield point of the material, the column begins to fail. Its load capacity is limited by the strength of the material.

In long, slender columns, however, failure may take place by buckling. This mode of instability is often sudden and can occur when the axial load in a column reaches a certain critical value. In many cases, the stress in the column may never reach the yield point. The load capacity of slender columns is not limited by the strength of the material but rather by the stiffness of the member.

Elastic buckling is a state of lateral instability that occurs while the material is stressed below the yield point. It is of special importance in structures with slender members.

A formula for the critical buckling load for pin-ended columns was derived by Euler in 1757 and is till in use. For the buckled shape under axial load P for a pin-ended column of constant cross section (Fig. 3.88*a*), Euler's column formula can be derived as follows:

With coordinate axes chosen as shown in Fig. 3.88*b*, moment equilibrium about one end of the column requires

$$M(x) + Py(x) = 0 \tag{3.161}$$

where $M(x)$ = bending moment at distance x from one end of the column
$y(x)$ = deflection of the column at distance x

Substitution of the moment-curvature relationship [Eq. (3.79)] into Eq. (3.161) gives

$$EI\frac{d^2y}{dx^2} + Py(x) = 0 \tag{3.162}$$

where E = modulus of elasticity of the material
I = moment of inertia of the cross section about the bending axis

The solution to this differential equation is

$$y(x) = A \cos \lambda x + B \sin \lambda x \tag{3.163}$$

where $\lambda = \sqrt{P/EI}$
A, B = unknown constants of integration

Substitution of the boundary condition $y(0) = 0$ into Eq. (3.163) indicates that $A = 0$. The additional boundary condition $y(L) = 0$ indicates that

$$B \sin \lambda L = 0 \tag{3.164}$$

where L is the length of the column. Equation (3.164) is often referred to as a **transcendental equation.** It indicates that either $B = 0$, which would be a trivial solution, or that λL must equal some multiple of π. The latter relationship provides the minimum critical value of P:

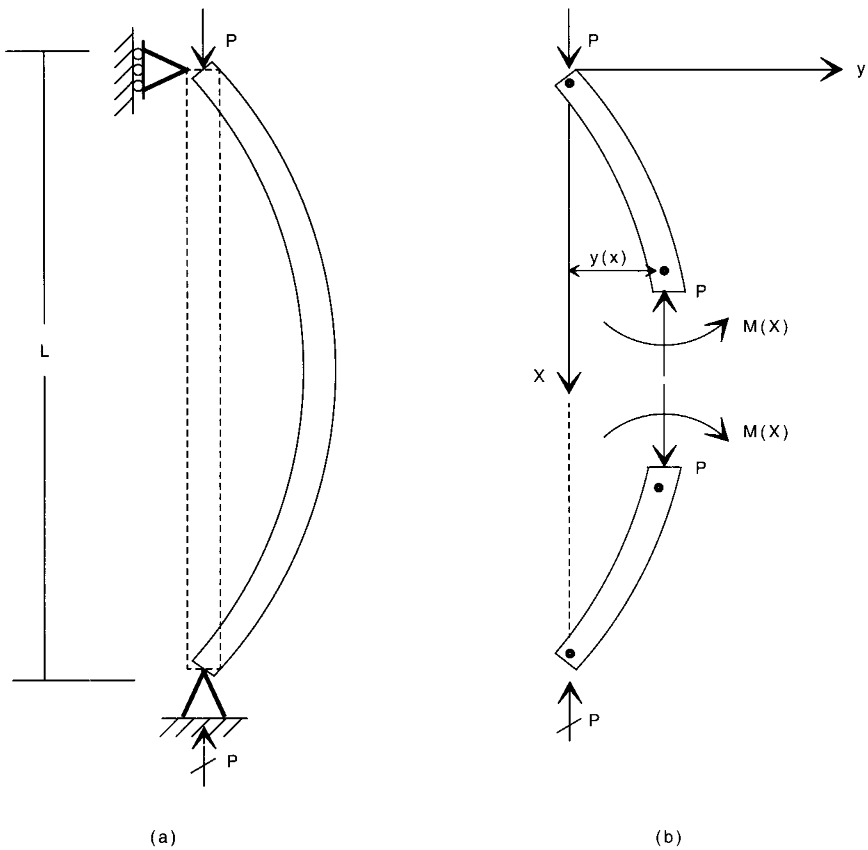


FIGURE 3.88 Buckling of a pin-ended column under axial load. (*b*) Internal forces hold the column in equilibrium.

$$\lambda L = \pi \qquad P = \frac{\pi^2 EI}{L^2} \tag{3.165}$$

This is the **Euler formula** for pin-ended columns. On substitution of Ar^2 for I, where A is the cross-sectional area and r the radius of gyration, Eq. (3.165) becomes

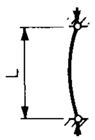
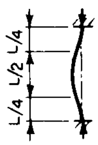
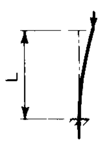
$$P = \frac{\pi^2 EA}{(L/r)^2} \tag{3.166}$$

L/r is called the **slenderness ratio** of the column.

Euler's formula applies only for columns that are perfectly straight, have a uniform cross section made of a linear elastic material, have end supports that are ideal pins, and are concentrically loaded.

Equations (3.165) and (3.166) may be modified to approximate the critical buckling load of columns that do not have ideal pins at the ends. Table 3.4 illustrates some ideal end conditions for slender columns and corresponding critical buckling loads. It indicates that elastic critical buckling loads may be obtained for all cases by substituting an **effective length** KL for the length L of the pinned column assumed for the derivation of Eq. (3.166):

TABLE 3.4 Buckling Formulas for Columns

Type of column	Effective length	Critical buckling load
	L	$\frac{\pi^2 EI}{L^2}$
	$\frac{L}{2}$	$\frac{4\pi^2 EI}{L^2}$
	$\approx 0.7L$	$\approx \frac{2\pi^2 EI}{L^2}$
	$2L$	$\frac{\pi^2 EI}{4L^2}$

$$P = \frac{\pi^2 EA}{(KL/r)^2} \tag{3.167}$$

Equation (3.167) also indicates that a column may buckle about either the section's major or minor axis depending on which has the greater slenderness ratio KL/r.

In some cases of columns with open sections, such as a cruciform section, the controlling buckling mode may be one of twisting instead of lateral deformation. If the warping rigidity of the section is negligible, **torsional buckling** in a pin-ended column will occur at an axial load of

$$P = \frac{GJA}{I_\rho} \tag{3.168}$$

where G = shear modulus of elasticity
J = torsional constant
A = cross-sectional area
I_ρ = polar moment of inertia = $I_x + I_y$

If the section possesses a significant amount of warping rigidity, the axial buckling load is increased to

$$P = \frac{A}{I_\rho}\left(GJ + \frac{\pi^2 EC_w}{L^2}\right) \tag{3.169}$$

where C_w is the warping constant, a function of cross-sectional shape and dimensions (see Fig. 3.89).

(S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability,* and F. Bleich, *Buckling Strength of Metal Structures,* McGraw-Hill, Inc., New York; T. V. Galambos, *Guide to Stability of Design of Metal Structures,* John Wiley & Sons, Inc. New York; W. McGuire, *Steel Structures,* Prentice-Hall, Inc., Englewood Cliffs, N.J.)

3.42 ELASTIC LATERAL BUCKLING OF BEAMS

Bending of the beam shown in Fig. 3.90*a* produces compressive stresses within the upper portion of the beam cross section and tensile stresses in the lower portion. Similar to the behavior of a column (Art. 3.41), a beam, although the compressive stresses may be well within the elastic range, can undergo lateral buckling failure. Unlike a column, however, the beam is also subjected to tension, which tends to restrain the member from lateral translation. Hence, when **lateral buckling** of the beam occurs, it is through a combination of twisting and out-of-plane bending (Fig. 3.90*b*).

For a simply supported beam of rectangular cross section subjected to uniform bending, buckling occurs at the critical bending moment

$$M_{cr} = \frac{\pi}{L}\sqrt{EI_y GJ} \tag{3.170}$$

where L = unbraced length of the member
E = modulus of elasticity
I_y = moment of inertial about minor axis
G = shear modulus of elasticity
J = torsional constant

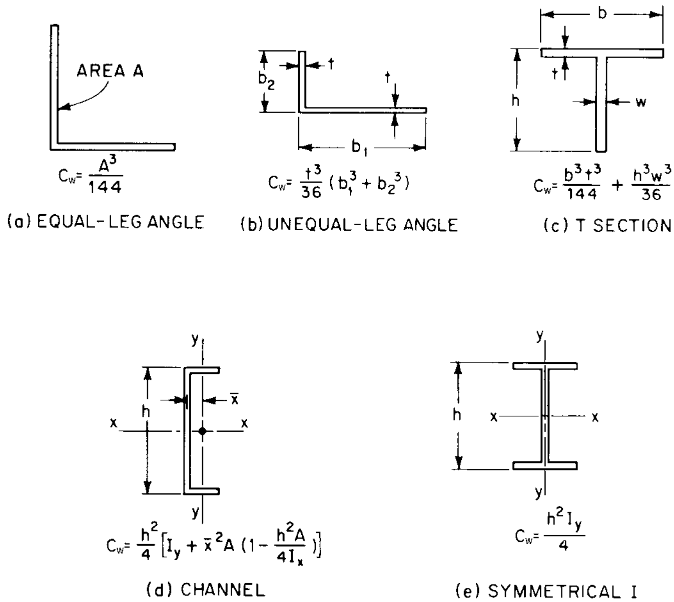


FIGURE 3.89 Torsion-bending constants for torsional buckling. A = cross-sectional area; I_x = moment of inertia about x–x axis; I_y = moment of inertia about y–y axis. (*After F. Bleich, Buckling Strength of Metal Structures, McGraw-Hill Inc., New York.*)

As indicted in Eq. (3.170), the critical moment is proportional to both the lateral bending stiffness EI_y/L and the torsional stiffness of the member GJ/L.

For the case of an open section, such as a wide-flange or I-beam section, warping rigidity can provide additional torsional stiffness. Buckling of a simply supported beam of open cross section subjected to uniform bending occurs at the critical bending moment

$$M_{cr} = \frac{\pi}{L}\sqrt{EI_y\left(GJ + EC_w\frac{\pi^2}{L^2}\right)} \tag{3.171}$$

where C_w is the warping constant, a function of cross-sectional shape and dimensions (see Fig. 3.89).

In Eq. (3.170) and (3.171), the distribution of bending moment is assumed to be uniform. For the case of a nonuniform bending-moment gradient, buckling often occurs at a larger critical moment. Approximation of this critical bending moment M'_{cr} may be obtained by multiplying M_{cr} given by Eq. (3.170) or (3.171) by an amplification factor:

$$M'_{cr} = C_b M_{cr} \tag{3.172}$$

where $C_b = \dfrac{12.5M_{max}}{2.5M_{max} + 3M_A + 4M_B + 3M_C}$ and (3.172*a*)

M_{max} = absolute value of maximum moment in the unbraced beam segment
M_A = absolute value of moment at quarter point of the unbraced beam segment
M_B = absolute value of moment at centerline of the unbraced beam segment
M_C = absolute value of moment at three-quarter point of the unbraced beam segment

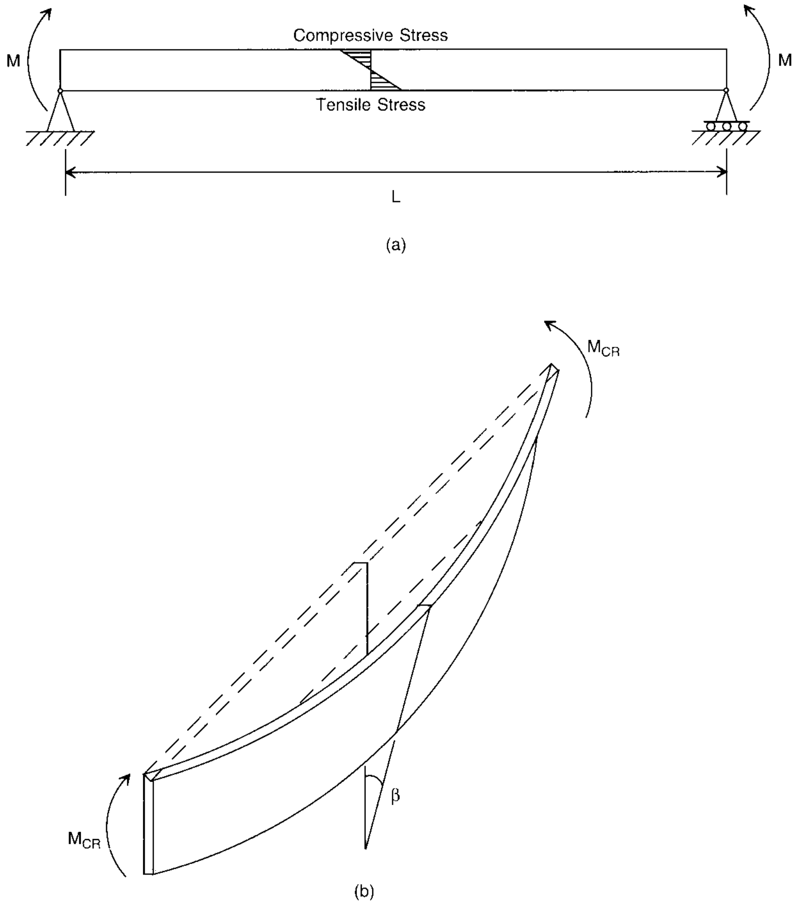


FIGURE 3.90 (*a*) Simple beam subjected to equal end moments. (*b*) Elastic lateral buckling of the beam.

C_b equals 1.0 for unbraced cantilevers and for members where the moment within a significant portion of the unbraced segment is greater than or equal to the larger of the segment end moments.

(S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability,* and F. Bleich, *Buckling Strength of Metal Structures,* McGraw-Hill, Inc., New York; T. V. Galambos, *Guide to Stability of Design of Metal Structures,* John Wiley & SOns, Inc., New York; W. McGuire, *Steel Structures,* Prentice-Hall, Inc., Englewood Cliffs, N.J.; *Load and Resistance Factor Design Specification for Structural Steel Buildings, American Institute of Steel Construction, Chicago, Ill.)*

3.43 ELASTIC FLEXURAL BUCKLING OF FRAMES

In Arts. 3.41 and 3.42, elastic instabilities of isolated columns and beams are discussed. Most structural members, however, are part of a structural system where the ends of the members are restrained by other members. In these cases, the instability of the system governs the critical buckling loads on the members. It is therefore important that frame behavior be incorporated into stability analyses. For details on such analyses, see T. V. Galambos, *Guide to Stability of Design of Metal Structures,* John Wiley & Sons, Inc., New York; S. Timoshenko and J. M. Gere, *Theory of Elastic Stability,* and F. Bleich, *Buckling Strength of Metal Structures,* McGraw-Hill, Inc., New York.

3.44 LOCAL BUCKLING

Buckling may sometimes occur in the form of wrinkles in thin elements such as webs, flanges, cover plates, and other parts that make up a section. This phenomenon is called **local buckling.**

The critical buckling stress in rectangular plates with various types of edge support and edge loading in the plane of the plates is given by

$$f_{cr} = k \frac{\pi^2 E}{12(1 - \mu^2)(b/t)^2} \tag{3.173}$$

where k = constant that depends on the nature of loading, length-to-width ratio of plate, and edge conditions
E = modulus of elasticity
μ = Poisson's ratio [Eq. (3.39)]
b = length of loaded edge of plate, or when the plate is subjected to shearing forces, the smaller lateral dimension
t = plate thickness

Table 3.5 lists values of k for various types of loads and edge support conditions. (From formulas, tables, and curves in F. Bleich, *Buckling Strength of Metal Structures,* S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability,* and G. Gernard. *Introduction to Structural Stability Theory,* McGraw-Hill, Inc., New York.)

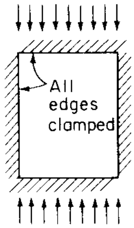
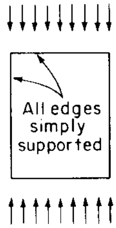
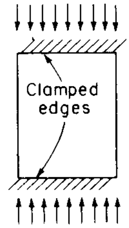
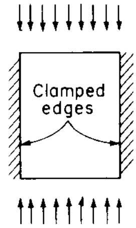
NONLINEAR BEHAVIOR OF STRUCTURAL SYSTEMS

Contemporary methods of steel design require engineers to consider the behavior of a structure as it reaches its limit of resistance. Unless premature failure occurs due to local buckling, fatigue, or brittle fracture, the strength limit-state behavior will most likely include a nonlinear response. As a frame is being loaded, **nonlinear behavior** can be attributed primarily to second-order effects associated with changes in geometry and yielding of members and connections.

3.45 COMPARISONS OF ELASTIC AND INELASTIC ANALYSES

In Fig. 3.91, the empirical limit-state response of a frame is compared with response curves generated in four different types of analyses: **first-order elastic analysis**, **second-order elastic analysis**, **first-order inelastic analysis**, and **second-order inelastic analysis**. In a first-order analysis, **geometric nonlinearities** are not included. These effects are accounted for, however, in a second-order analysis. **Material nonlinear** behavior is not included in an elastic analysis but is incorporated in an inelastic analysis.

TABLE 3.5 Values of k for Buckling Stress in Thin Plates

$\frac{a}{b}$	 Case 1	 Case 2	 Case 3	 Case 4
0.4	28.3	8.4	9.4	
0.6	15.2	5.1	13.4	7.1
0.8	11.3	4.2	8.7	7.3
1.0	10.1	4.0	6.7	7.7
1.2	9.4	4.1	5.8	7.1
1.4	8.7	4.5	5.5	7.0
1.6	8.2	4.2	5.3	7.3
1.8	8.1	4.0	5.2	7.2
2.0	7.9	4.0	4.9	7.0
2.5	7.6	4.1	4.5	7.1
3.0	7.4	4.0	4.4	7.1
3.5	7.3	4.1	4.3	7.0
4.0	7.2	4.0	4.2	7.0
∞	7.0	4.0	4.0	∞

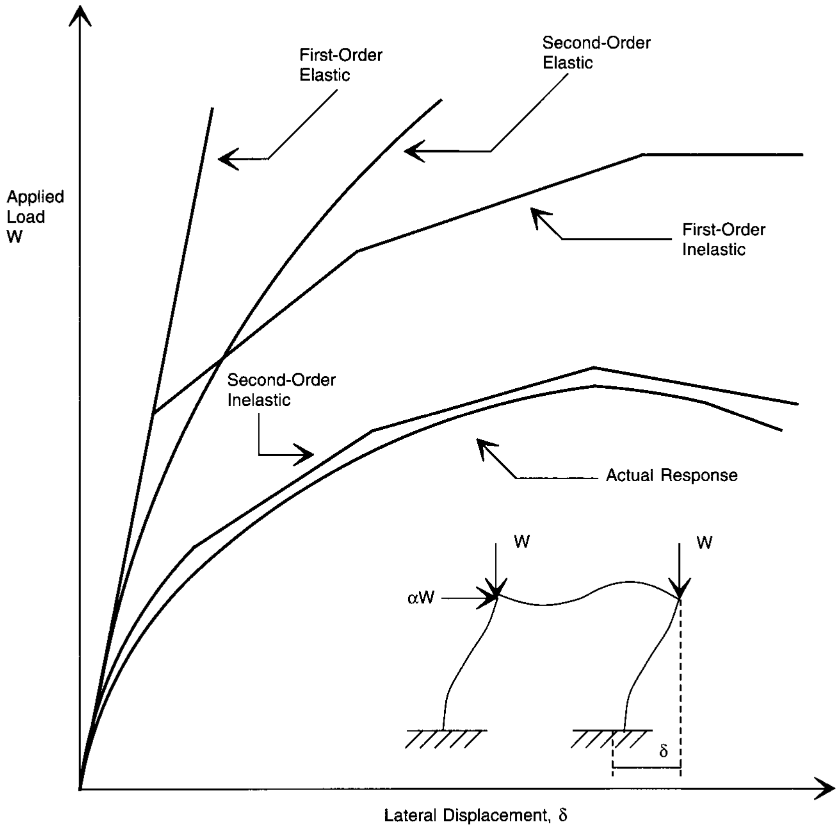


FIGURE 3.91 Load-displacement responses for a rigid frame determined by different methods of analysis.

In most cases, second-order and inelastic effects have interdependent influences on frame stability; i.e., second-order effects can lead to more inelastic behavior, which can further amplify the second-order effects. Producing designs that account for these nonlinearities requires use of either conventional methods of linear elastic analysis (Arts. 3.29 to 3.39) supplemented by semiempirical or judgmental allowances for nonlinearity or more advanced methods of nonlinear analysis.

3.46 GENERAL SECOND-ORDER EFFECTS

A column unrestrained at one end with length L and subjected to horizontal load H and vertical load P (Fig. 3.92*a*) can be used to illustrate the general concepts of second-order behavior. If E is the modulus of elasticity of the column material and I is the moment of inertia of the column, and the equations of equilibrium are formulated on the undeformed geometry, the first-order deflection at the top of the column is $\Delta_1 = HL^3/3EI$, and the first-order moment at the base of the column is $M_1 = HL$ (Fig. 3.92*b*). As the column deforms, however, the applied loads move with the top of the column through a deflection δ. In this

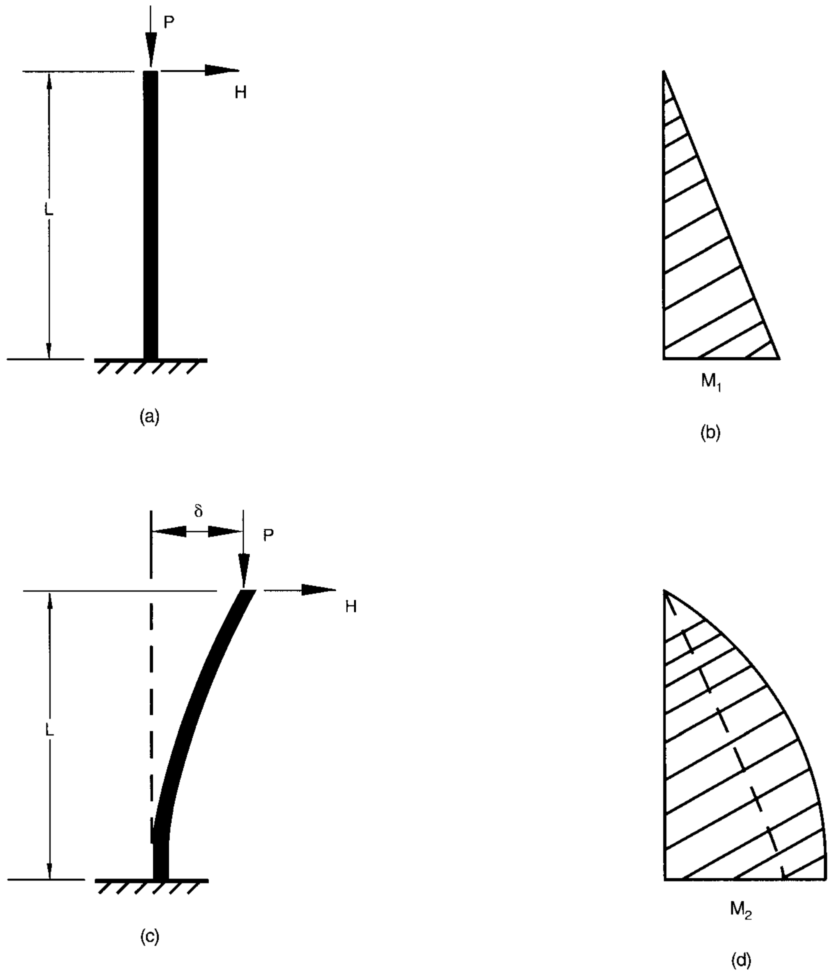


FIGURE 3.92 (*a*) Column unrestrained at one end, where horizontal and vertical loads act. (*b*) First-order maximum bending moment M_1 occurs at the base. (*c*) The column with top displaced by the forces. (*d*) Second-order maximum moment M_2 occurs at the base.

case, the actual second-order deflection $\delta = \Delta_2$ not only includes the deflection due to the horizontal load H but also the deflection due to the eccentricity generated with respect to the neutral axis of the column when the vertical load P is displaced (Fig. 3.92*c*). From equations of equilibrium for the deformed geometry, the second-order base moment is $M_2 = HL + P\Delta_2$ (Fig. 3.92*d*). The additional deflection and moment generated are examples of second-order effects or geometric nonlinearities.

In a more complex structure, the same type of second-order effects can be present. They may be attributed primarily to two factors: the axial force in a member having a significant influence on the bending stiffness of the member and the relative lateral displacement at the ends of members. Where it is essential that these destabilizing effects are incorporated within a limit-state design procedure, general methods are presented in Arts. 3.47 and 3.48.

3.47 APPROXIMATE AMPLIFICATION FACTORS FOR SECOND-ORDER EFFECTS

One method for approximating the influences of second-order effects (Art. 3.46) is through the use of **amplification factors** that are applied to first-order moments. Two factors are typically used. The first factor accounts for the additional deflection and moment produced by a combination of compressive axial force and lateral deflection δ along the span of a member. It is assumed that there is no relative lateral translation between the two ends of the member. The additional moment is often termed the ***P*δ moment**. For a member subject to a uniform first-order bending moment M_{nt} and axial force P (Fig. 3.93) with no relative translation of the ends of the member, the amplification factor is

$$B_1 = \frac{1}{1 - P/P_e} \tag{3.174}$$

where P_e is the elastic critical buckling load about the axis of bending (see Art. 3.41). Hence the moments from a second-order analysis when no relative translation of the ends of the member occurs may be approximated by

$$M_{2nt} = B_1 M_{nt} \tag{3.175}$$

where $B_1 \geq 1$.

The amplification factor in Eq. (3.174) may be modified to account for a non-uniform moment or moment gradient (Fig. 3.94) along the span of the member:

$$B_1 = \frac{C_m}{1 - P/P_e} \tag{3.176}$$

where C_m is a coefficient whose value is to be taken as follows:

1. For compression members with ends restrained from joint translation and not subject to transverse loading between supports, $C_m = 0.6 - 0.4(M_1/M_2)$, M_1 is the smaller and M_2 is the larger end moment in the unbraced length of the member. M_1/M_2 is positive when the moments cause reverse curvature and negative when they cause single curvature.

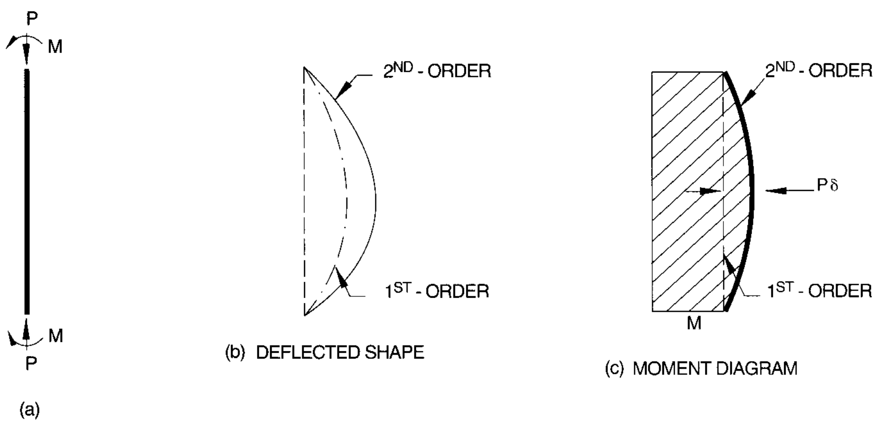


FIGURE 3.93 *P*δ effect for beam-column with uniform bending.

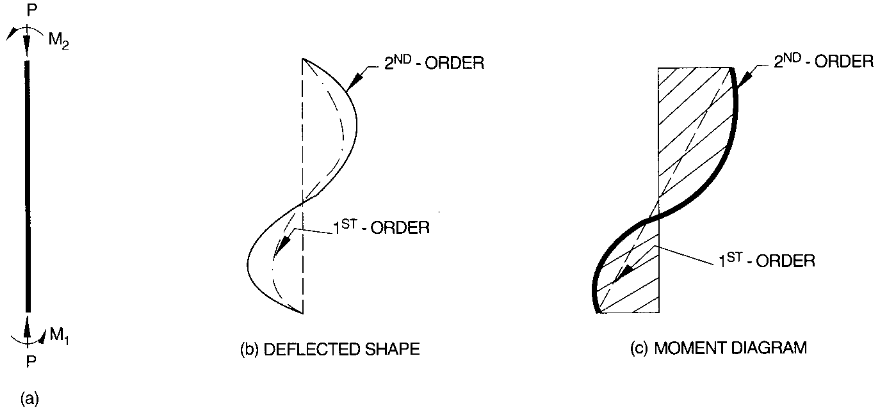


FIGURE 3.94 *$P\delta$ effect for beam-column with nonuniform bending.*

2. For compression members subject to transverse loading between supports, $C_m = 1.0$.

The second amplification factor accounts for the additional deflections and moments that are produced in a frame that is subject to sidesway, or drift. By combination of compressive axial forces and relative lateral translation of the ends of members, additional moments are developed. These moments are often termed the ***P*Δ moments.** In this case, the moments M_{lt} determined from a first-order analysis are amplified by the factor

$$B_2 = \frac{1}{1 - \dfrac{\Sigma P}{\Sigma P_e}} \tag{3.177}$$

where ΣP = total axial load of all columns in a story
ΣP_e = sum of the elastic critical buckling loads about the axis of bending for all columns in a story

Hence the moments from a second-order analysis when lateral translation of the ends of the member occurs may be approximated by

$$M_{2lt} = B_2 M_{lt} \tag{3.178}$$

For an unbraced frame subjected to both horizontal and vertical loads, both $P\delta$ and $P\Delta$ second-order destabilizing effects may be present. To account for these effects with amplification factors, two first-order analyses are required. In the first analysis, *nt* (no translation) moments are obtained by applying only vertical loads while the frame is restrained from lateral translation. In the second analysis, *lt* (liner translation) moments are obtained for the given lateral loads and the restraining lateral forces resulting from the first analysis. The moments from an actual second-order analysis may then be approximated by

$$M = B_1 M_{nt} + B_2 M_{lt} \tag{3.179}$$

(T. V. Galambos, *Guide to Stability of Design of Metal Structures,* John Wiley & Sons, Inc, New York; W. McGuire, *Steel Structures,* Prentice-Hall, Inc., Englewood Cliffs, N.J.; *Load and Resistance Factor Design Specifications for Structural Steel Buildings,* American Institute of Steel Construction, Chicago, Ill.)

3.48 GEOMETRIC STIFFNESS MATRIX METHOD FOR SECOND-ORDER EFFECTS

The conventional matrix stiffness method of analysis (Art. 3.39) may be modified to include directly the influences of second-order effects described in Art. 3.46. When the response of the structure is nonlinear, however, the linear relationship in Eq. (3.145), $\mathbf{P} = \mathbf{K\Delta}$, can no longer be used. An alternative is a numerical solution obtained through a sequence of linear steps. In each step, a load increment is applied to the structure and the stiffness and geometry of the frame are modified to reflect its current loaded and deformed state. Hence Eq. (3.145) is modified to the incremental form

$$\boldsymbol{\delta}\mathbf{P} = \mathbf{K_t}\boldsymbol{\delta\Delta} \tag{3.180}$$

where $\boldsymbol{\delta}\mathbf{P}$ = the applied load increment
$\mathbf{K_t}$ = the modified or tangent stiffness matrix of the structure
$\boldsymbol{\delta\Delta}$ = the resulting increment in deflections

The tangent stiffness matrix $\mathbf{K_t}$ is generated from nonlinear member force-displacement relationships. They are reflected by the nonlinear member stiffness matrix

$$\mathbf{k}' = \mathbf{k}'_E + \mathbf{k}'_G \tag{3.181}$$

where $\mathbf{k}'_E$ = the conventional elastic stiffness matrix (Art. 3.39)
$\mathbf{k}'_G$ = a geometric stiffness matrix which depends not only on geometry but also on the existing internal member forces.

In this way, the analysis ensures that the equations of equilibrium are sequentially being formulated for the deformed geometry and that the bending stiffness of all members is modified to account for the presence of axial forces.

Inasmuch as a piecewise linear procedure is used to predict nonlinear behavior, accuracy of the analysis increases as the number of load increments increases. In many cases, however, good approximations of the true behavior may be found with relatively large load increments. The accuracy of the analysis may be confirmed by comparing results with an additional analysis that uses smaller load steps.

(W. McGuire, R. H. Gallagher, and R. D. Ziemian, *Matrix Structural Analysis,* John Wiley & Sons, Inc., New York; W. F. Chen and E. M. Lui, *Stability Design of Steel Frames,* CRC Press, Inc., Boca Raton, Fla.; T. V. Galambos, *Guide to Stability Design Criteria for Metal Structures,* John Wiley & Sons, Inc., New York)

3.49 GENERAL MATERIAL NONLINEAR EFFECTS

Most structural steels can undergo large deformations before rupturing. For example, yielding in ASTM A36 steel begins at a strain of about 0.0012 in per in and continues until strain hardening occurs at a strain of about 0.014 in per in. At rupture, strains can be on the order of 0.25 in per in. These material characteristics affect the behavior of steel members strained into the yielding range and form the basis for the plastic theory of analysis and design.

The **plastic capacity** of members is defined by the amount of axial force and bending moment required to completely yield a member's cross section. In the absence of bending, the plastic capacity of a section is represented by the **axial yield load**

$$P_y = AF_y \tag{3.182}$$

where A = cross-sectional area
F_y = yield stress of the material

For the case of flexure and no axial force, the plastic capacity of the section is defined by the **plastic moment**

$$M_p = ZF_y \tag{3.183}$$

where Z is the plastic section modulus (Art. 3.16). The plastic moment of a section can be significantly greater than the moment required to develop first yielding in the section, defined as the **yield moment**

$$M_y = SF_y \tag{3.184}$$

where S is the elastic section modulus (Art. 3.16). The ratio of the plastic modulus to the elastic section modulus is defined as a section's **shape factor**

$$s = \frac{Z}{S} \tag{3.185}$$

The shape factor indicates the additional moment beyond initial yielding that a section can develop before becoming completely yielded. The shape factor ranges from about 1.1 for wide-flange sections to 1.5 for rectangular shapes and 1.7 for round sections.

For members subjected to a combination of axial force and bending, the plastic capacity of the section is a function of the section geometry. For example, one estimate of the plastic capacity of a wide-flange section subjected to an axial force P and a major-axis bending moment M_{xx} is defined by the interaction equation

$$\frac{P}{P_y} + 0.85\frac{M_{xx}}{M_{px}} = 1.0 \tag{3.186}$$

where M_{px} = major-axis plastic moment capacity = $Z_{xx}F_y$. An estimate of the minor-axis plastic capacity of wide-flange section is

$$\left(\frac{P}{P_y}\right)^2 + 0.84\frac{M_{yy}}{M_{py}} = 1.0 \tag{3.187}$$

where M_{yy} = minor-axis bending moment, and M_{py} = minor-axis plastic moment capacity = Z_yF_y.

When one section of a member develops its plastic capacity, an increase in load can produce a large rotation or axial deformation or both, at this location. When a large rotation occurs, the fully yielded section forms a **plastic hinge.** It differs from a true hinge in that some deformation remains in a plastic hinge after it is unloaded.

The plastic capacity of a section may differ from the ultimate strength of the member or the structure in which it exists. First, if the member is part of a redundant system (Art. 3.28), the structure can sustain additional load by distributing the corresponding effects away from the plastic hinge and to the remaining unyielded portions of the structure. Means for accounting for this behavior are incorporated into inelastic methods of analysis.

Secondly, there is a range of strain hardening beyond F_y that corresponds to large strains but in which a steel member can develop an increased resistance to additional loads. This assumes, however, that the section is adequately braced and proportioned so that local or lateral buckling does not occur.

Material nonlinear behavior can be demonstrated by considering a simply supported beam with span $L = 400$ in and subjected to a uniform load w (Fig. 3.95*a*). The maximum moment at midspan is $M_{max} = wL^2/8$ (Fig. 3.95*b*). If the beam is made of a $W24 \times 103$ wide-flange

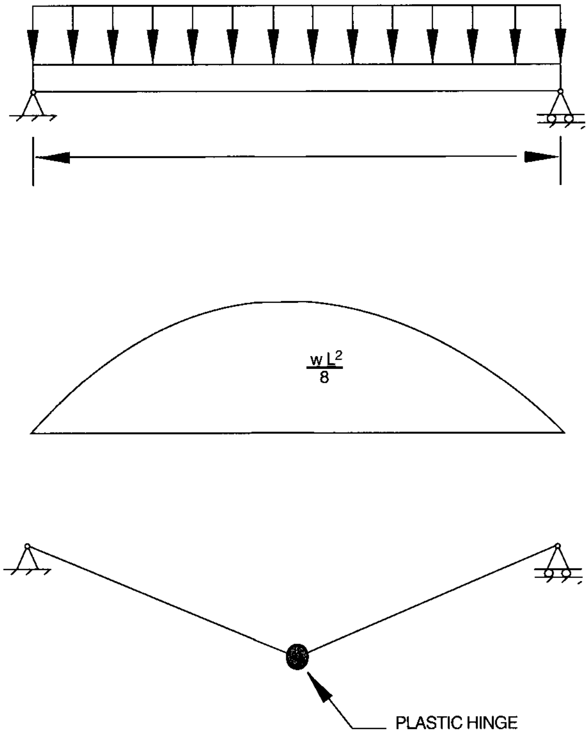


FIGURE 3.95 (*a*) Uniformly loaded simple beam. (*b*) Moment diagram. (*c*) Development of a plastic hinge at midspan.

section with a yield stress $F_y = 36$ ksi and a section modulus $S_{xx} = 245$ in^3, the beam will begin to yield at a bending moment of $M_y = F_y S_{xx} = 36 \times 245 = 8820$ in-kips. Hence, when beam weight is ignored, the beam carries a uniform load $w = 8M_y/L^2 = 8 \times 8820/400^2 = 0.44$ kips/in.

A $W24 \times 103$ shape, however; has a plastic section modulus $Z_{xx} = 280$ in^3. Consequently, the plastic moment equals $M_p = F_y Z_{xx} = 36 \times 280 = 10{,}080$ in-kips. When beam weight is ignored, this moment is produced by a uniform load $w = 8M_p/L^2 = 8 \times 10{,}080/400^2 = 0.50$ kips/in, an increase of 14% over the load at initiation of yield. The load developing the plastic moment is often called the **limit,** or **ultimate load.** It is under this load that the beam, with hinges at each of its supports, develops a plastic hinge at midspan (Fig. 3.95*c*) and becomes unstable. If strain-hardening effects are neglected, a kinematic mechanism has formed, and no further loading can be resisted.

If the ends of a beam are fixed as shown in Fig. 3.96*a*, the midspan moment is $M_{\mathrm{mid}} = wL^2/24$. The maximum moment occur at the ends, $M_{\mathrm{end}} = wL^2/12$ (Fig. 3.96*b*). If the beam has the same dimensions as the one in Fig. 3.95*a*, the beam begins to yield at uniform load $w = 12M_y/L^2 = 12 \times 8820/400^2 = 0.66$ kips/in. If additional load is applied to the beam, plastic hinges eventually form at the ends of the beam at load $w = 12M_p/L^2 = 12 \times 10{,}080/400^2 = 0.76$ kips/in. Although plastic hinges exist at the supports, the beam is still stable at this load. Under additional loading, it behaves as a simply supported beam with moments M_p at each end (Fig. 3.96*c*) and a maximum moment $M_{\mathrm{mid}} = wL^2/8 - M_p$ at midspan (Fig. 3.96*d*). The limit load of the beam is reached when a plastic hinge forms at midspan, $M_{\mathrm{mid}} = M_p$, thus creating a mechanism (Fig. 3.96*e*). The uniform load at which this occurs

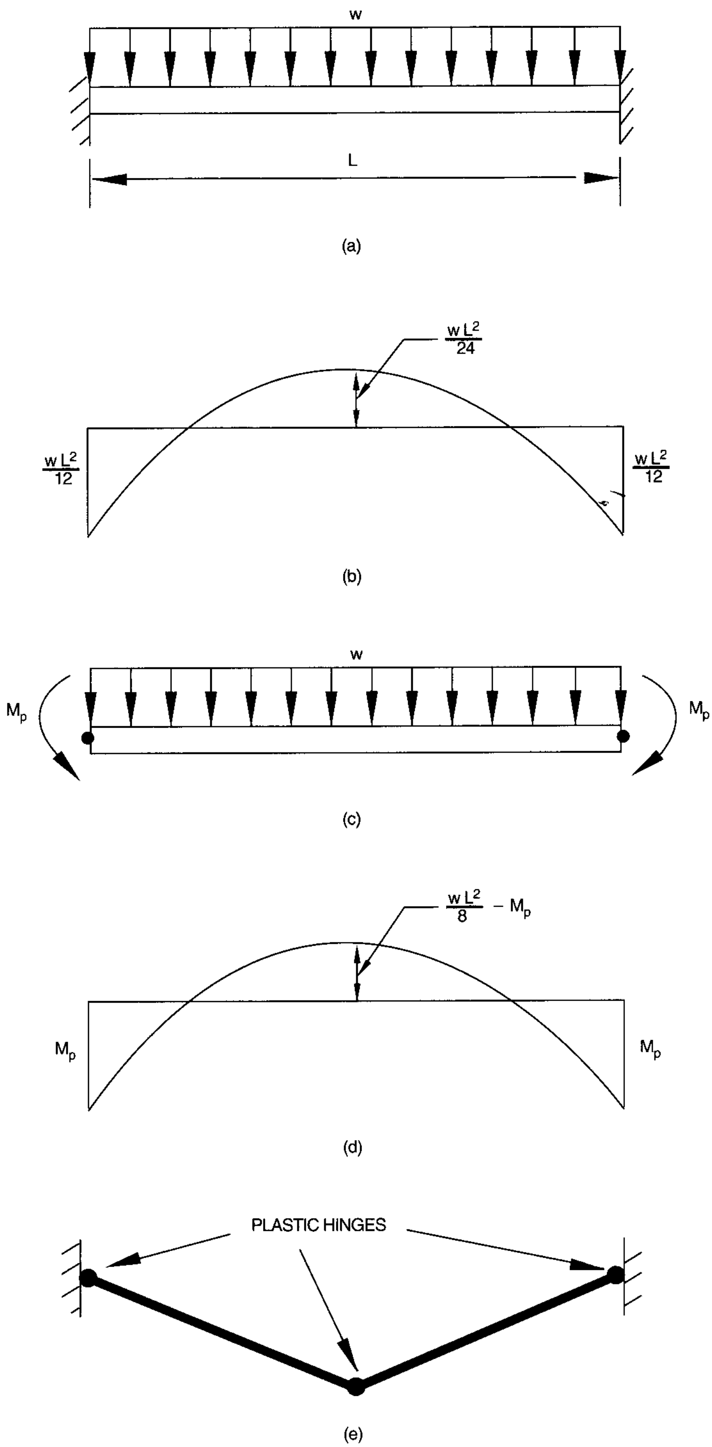


FIGURE 3.96 (*a*) Uniformly loaded fixed-end beam. (*b*) Moment diagram. (*c*) Beam with plastic hinges at both ends. (*d*) Moment diagram for the plastic condition. (*e*) Beam becomes unstable when plastic hinge also develops in the interior.

is $w = 2M_p \times 8/L^2 = 2 \times 10{,}080 \times 8/400^2 = 1.01$ kips/in, a load that is 53% greater than the load at which initiation of yield occurs and 33% greater than the load that produces the first plastic hinges.

3.50 CLASSICAL METHODS OF PLASTIC ANALYSIS

In continuous structural systems with many members, there are several ways that mechanisms can develop. The **limit load,** or **load creating a mechanism,** lies between the loads computed from upper-bound and lower-bound theorems. The **upper-bound theorem** states that a load computed on the basis of an assumed mechanism will be greater than, or at best equal to, the true limit load. The **lower-bound theorem** states that a load computed on the basis of an assumed bending-moment distribution satisfying equilibrium conditions, with bending moments nowhere exceeding the plastic moment M_p, is less than, or at best equal to, the true limit load. The plastic moment is $M_p = ZF_y$, where Z = plastic section modulus and F_y = yield stress. If both theorems yield the same load, it is the true ultimate load.

In the application of either theorem, the following conditions must be satisfied at the limit load: External forces must be in equilibrium with internal forces; there must be enough plastic hinges to form a mechanism; and the plastic moment must not be exceeded anywhere in the structure.

The process of investigating mechanism failure loads to determine the maximum load a continuous structure can sustain is called **plastic analysis.**

3.50.1 Equilibrium Method

The **statical** or **equilibrium method** is based on the lower-bound theorem. It is convenient for continuous structures with few members. The steps are

- Select and remove redundants to make the structure statically determinate.
- Draw the moment diagram for the given loads on the statically determinate structure.
- Sketch the moment diagram that results when an arbitrary value of each redundant is applied to the statically determinate structure.
- Superimpose the moment diagrams in such a way that the structure becomes a mechanism because there are a sufficient number of the peak moments that can be set equal to the plastic moment M_p.
- Compute the ultimate load from equilibrium equations.
- Check to see that M_p is not exceeded anywhere.

To demonstrate the method, a plastic analysis will be made for the two-span continuous beam shown in Fig. 3.97*a*. The moment at *C* is chosen as the redundant. Figure 3.97*b* shows the bending-moment diagram for a simple support at *C* and the moment diagram for an assumed redundant moment at *C*. Figure 3.97*c* shows the combined moment diagram. Since the moment at *D* appears to exceed the moment at *B*, the combined moment diagram may be adjusted so that the right span becomes a mechanism when the peak moments at *C* and *D* equal the plastic moment M_p (Fig. 3.97*d*).

If $M_C = M_D = M_p$, then equilibrium of span *CE* requires that at *D*,

$$M_p = \frac{2L}{3}\left(\frac{2P}{3} - \frac{M_p}{L}\right) = \frac{4PL}{9} - \frac{2M_p}{3}$$

from which the ultimate load P_u may be determined as

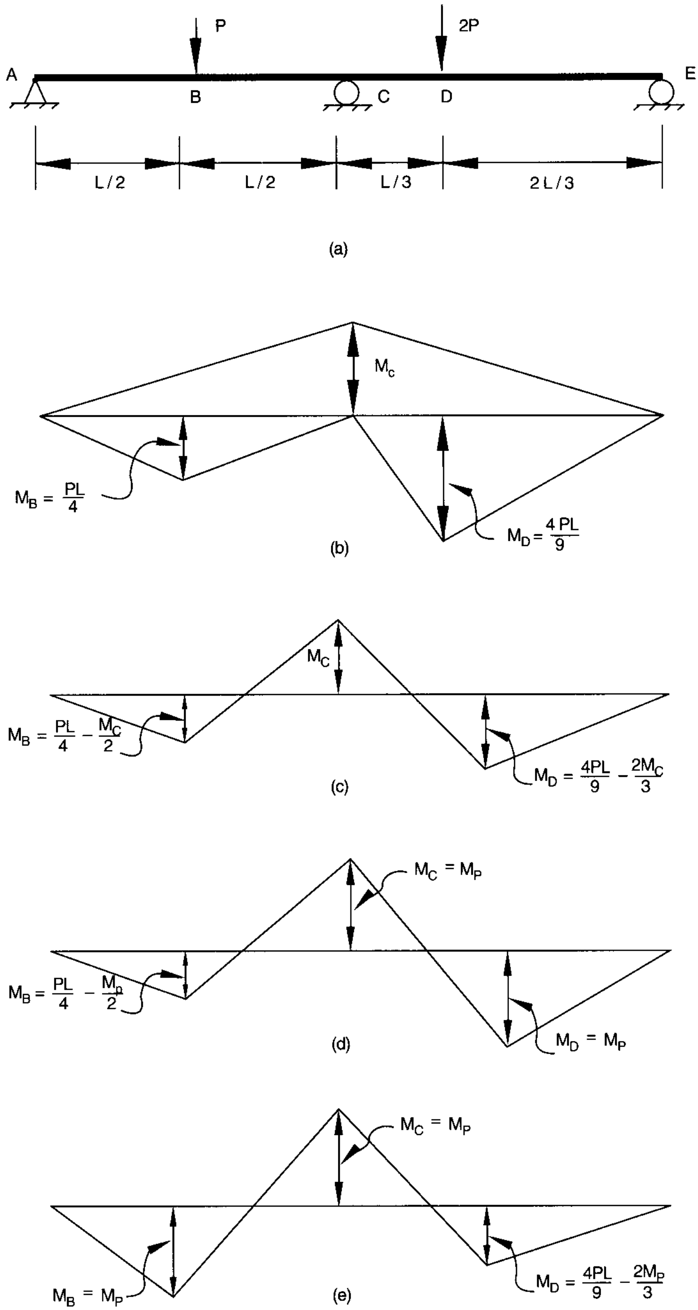


FIGURE 3.97 (*a*) Two-span continuous beam with concentrated loads. (*b*) Moment diagrams for positive and negative moments. (*c*) Combination of moment diagrams in (*b*). (*d*) Valid solution for ultimate load is obtained with plastic moments at peaks at *C* and *D*, and M_p is not exceeded anywhere. (*e*) Invalid solution results when plastic moment is assumed to occur at *B* and M_p is exceeded at *D*.

$$P_u = \frac{9}{4L}\left(M_p + \frac{2M_p}{3}\right) = \frac{15M_p}{4L}$$

The peak moment at B should be checked to ensure that $M_B \leq M_p$. For the ultimate load P_u and equilibrium in span AC,

$$M_B = \frac{P_u L}{4} - \frac{M_p}{2} = \frac{15M_p}{4L}\frac{L}{4} - \frac{M_p}{2} = \frac{7M_p}{16} < M_p$$

This indicates that at the limit load, a plastic hinge will not form in the center of span AC.

If the combined moment diagram had been adjusted so that span AC becomes a mechanism with the peak moments at B and C equaling M_p (Fig. 3.97*e*), this would not be a statically admissible mode of failure. Equilibrium of span AC requires

$$P_u = \frac{6M_p}{L}$$

Based on equilibrium of span CE, this ultimate load would cause the peak moment at D to be

$$M_D = \frac{4P_u L}{9} - \frac{2M_p}{3} = \frac{6M_p}{L}\frac{4L}{9} - \frac{2M_p}{3} = 2M_p$$

In this case, M_D violates the requirement that M_p cannot be exceeded. The moment diagram in Fig. 3.97*e* is not valid.

3.50.2 Mechanism Method

As an alternative, the **mechanism method** is based on the upper-bound theorem. It includes the following steps:

- Determine the locations of possible plastic hinges.
- Select plastic-hinge configurations that represent all possible mechanism modes of failure.
- Using the principle of virtual work, which equates internal work to external work, calculate the ultimate load for each mechanism.
- Assume that the mechanism with the lowest critical load is the most probable and hence represents the ultimate load.
- Check to see that M_p is not exceeded anywhere.

To illustrate the method, the ultimate load will be found for the continuous beam in Fig. 3.97*a*. Basically, the beam will become unstable when plastic hinges form at B and C (Fig. 3.98*a*) or C and D (Fig. 3.98*b*). The resulting constructions are called either **independent** or **fundamental mechanisms.** The beam is also unstable when hinges form at B, C, and D (Fig. 3.98*c*). This configuration is called a **composite** or **combination mechanism** and also will be discussed.

Applying the principle of virtual work (Art. 3.23) to the beam mechanism in span AC (Fig. 3.98*d*), external work equated to internal work for a virtual end rotation θ gives

$$\theta P \frac{L}{2} = 2\theta M_p + \theta M_p$$

from which $P = 6M_p/L$.

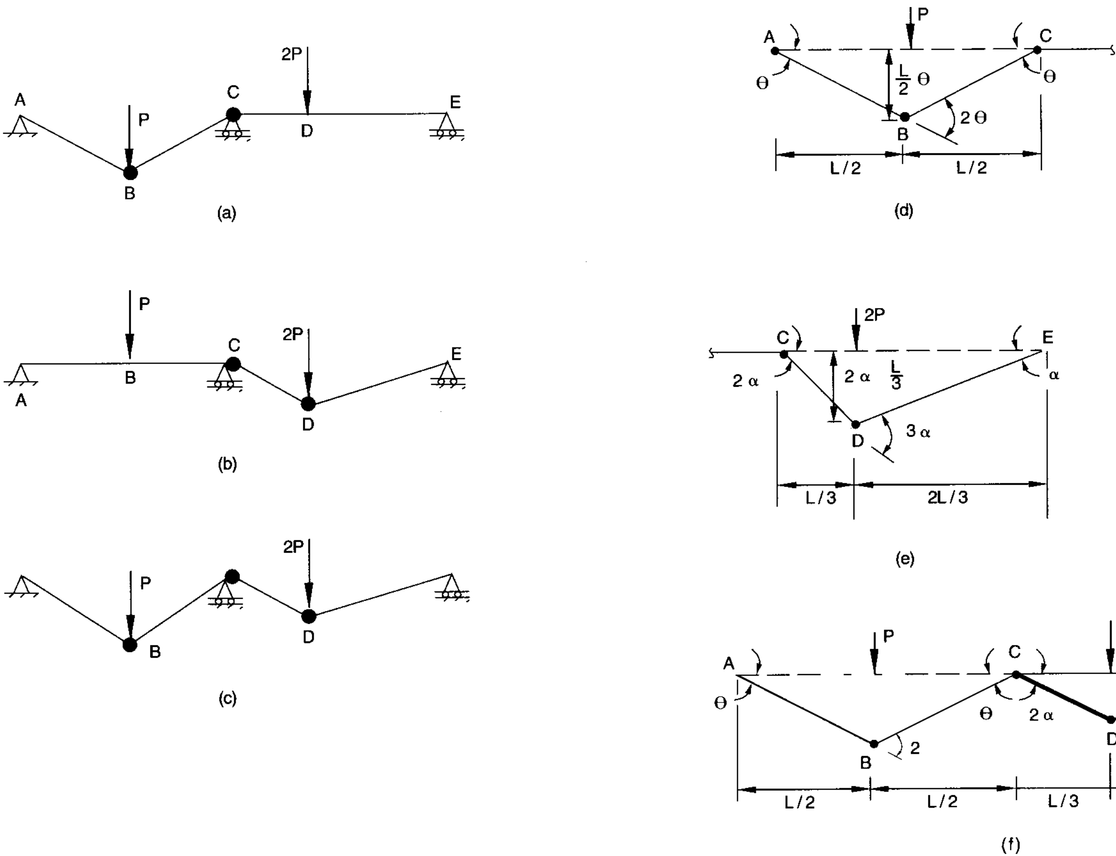


FIGURE 3.98 Plastic analysis of two-span continuous beam by the mechanism method. Beam mechanisms form when plastic hinges occur at (*a*) *B* and *C*, (*b*) *C* and *D*, and (*c*) *B*, *C*, and *D*. (*d*), (*e*) (*f*) show virtual displacements assumed for the mechanisms in (*a*), (*b*), and (*c*), respectively.

Similarly, by assuming a virtual end rotation α at E, a beam mechanism in span CE (Fig. 3.98*e*) yields

$$2P \frac{L}{3} 2\alpha = 2\alpha M_p + 3\alpha M_p$$

from which $P = 15M_p/4L$.

Of the two independent mechanisms, the latter has the lower critical load. This suggests that the ultimate load is $P_u = 15M_p/4L$.

For the combination mechanism (Fig. 3.98*f*), application of virtual work yields

$$\theta P \frac{L}{2} + 2P \frac{L}{3} 2\alpha = 2\theta M_p + \theta M_p + 2\alpha M_p + 3\alpha M_p$$

from which

$$P = (6M_p/L)[3(\theta/\alpha) + 5]/[3(\theta/\alpha) + 8]$$

In this case, the ultimate load is a function of the value assumed for the ratio θ/α. If θ/α equals zero, the ultimate load is $P = 15M_p/4L$ (the ultimate load for span CE as an independent mechanism). The limit load as θ/α approaches infinity is $P = 6M_p/L$ (the ultimate load for span AC as an independent mechanism). For all positive values of θ/α, this equation predicts an ultimate load P such that $15M_p/4L \leq P \leq 6M_pL$. This indicates that for a mechanism to form span AC, a mechanism in span CE must have formed previously. Hence the ultimate load for the continuous beam is controlled by the load required to form a mechanism in span CE.

In general, it is useful to determine all possible independent mechanisms from which composite mechanisms may be generated. The number of possible independent mechanisms m may be determined from

$$m = p - r \tag{3.188}$$

where p = the number of possible plastic hinges and r = the number of redundancies. Composite mechanisms are selected in such a way as to maximize the total external work or minimize the total internal work to obtain the lowest critical load. Composite mechanisms that include the displacement of several loads and elimination of plastic hinges usually provide the lowest critical loads.

3.50.3 Extension of Classical Plastic Analysis

The methods of plastic analysis presented in Secs. 3.50.1 and 3.50.2 can be extended to analysis of frames and trusses. However, such analyses can become complex, especially if they incorporate second-order effects (Art. 3.46) or reduction in plastic-moment capacity for members subjected to axial force and bending (Art. 3.49).

(E. H. Gaylord, Jr., et al., *Design of Steel Structures,* McGraw-Hill, Inc., New York; W. Prager, *An Introduction to Plasticity,* Addison-Wesley Publishing Company, Inc., Reading, Mass., L. S. Beedle, *Plastic Design of Steel Frames,* John Wiley & Sons; Inc., New York: *Plastic Design in Steel—A Guide and Commentary,* Manual and Report No. 41, American Society of Civil Engineers; R. O. Disque, *Applied Plastic Design in Steel,* Van Nostrand Reinhold Company, New York.)

3.51 CONTEMPORARY METHODS OF INELASTIC ANALYSIS

Just as the conventional matrix stiffness method of analysis (Art. 3.39) may be modified to directly include the influences of second-order effects (Art. 3.48), it also may be modified to incorporate nonlinear behavior of structural materials. Loads may be applied in increments to a structure and the stiffness and geometry of the frame changed to reflect its current deformed and possibly yielded state. The tangent stiffness matrix $\mathbf{K_t}$ in Eq. (3.180) is generated from nonlinear member force-displacement relationships. To incorporate material nonlinear behavior, these relationships may be represented by the nonlinear member stiffness matrix

$$\mathbf{k}' = \mathbf{k}'_E + \mathbf{k}'_G + \mathbf{k}'_p \tag{3.189}$$

where $\mathbf{k}'_E$ = the conventional elastic stiffness matrix (Art. 3.39)
$\mathbf{k}'_G$ = a geometric stiffness matrix (Art. 3.48)
$\mathbf{k}'_P$ = a plastic reduction stiffness matrix that depends on the existing internal member forces.

In this way, the analysis not only accounts for second-order effects but also can directly account for the destabilizing effects of material nonlinearities.

In general, there are two basic inelastic stiffness methods for investigating frames: the **plastic-zone** or **spread of plasticity method** and the **plastic-hinge** or **concentrated plasticity method.** In the plastic-zone method, yielding is modeled throughout a member's volume, and residual stresses and material strain-hardening effects can be included directly in the analysis. In a plastic-hinge analysis, material nonlinear behavior is modeled by the formation of plastic hinges at member ends. Hinge formation and any corresponding plastic deformations are controlled by a **yield surface,** which may incorporate the interaction of axial force and biaxial bending.

(T. V. Galambos, *Guide to Stability Design Criteria for Metal Structures,* John Wiley & Sons, New York; W. F. Chen and E. M. Lui, *Stability Design of Steel Frames,* CRC Press, Inc., Boca Raton, Fla.; and W. McGuire, R. H. Gallagher, and R. D. Ziemian, *Matrix Structural Analysis,* John Wiley & Sons, Inc., New York.)

TRANSIENT LOADING

Dynamic loads are one of the types of loads to which structures may be subjected (Art. 3.26). When dynamic effects are insignificant, they usually are taken into account in design by application of an impact factor or an increased factor of safety. In many cases, however, an accurate analysis based on the principles of dynamics is necessary. Such an analysis is paticularly desirable when a structure is acted on by unusually strong wind gusts, earthquake shocks, or impulsive loads, such as blasts.

3.52 GENERAL CONCEPTS OF STRUCTURAL DYNAMICS

There are many types of dynamic loads. **Periodic loads** vary cyclically with time. **Nonperiodic loads** do not have a specific pattern of variation with time. **Impulsive dynamic loading** is independent of the motion of the structure. **Impactive dynamic loading** includes the interaction of all external and internal forces and thus depends on the motions of the structure and of the applied load.

To define a loading within the context of a dynamic or transient analysis, one must specify the direction and magnitude of the loading at every instant of time. The loading may come from either time-dependent forces being applied directly to the structure or from time-dependent motion of the structure's supports, such as a steel frame subjected to earthquake loading.

The term **response** is often used to describe the effects of dynamic loads on structures. More specifically, a response to dynamic loads may represent the displacement, velocity, or acceleration at any point within a structure over a duration of time.

A reciprocating or oscillating motion of a body is called **vibration.** If vibration takes place in the absence of external forces but is accompanied by external or internal frictional forces, or both, it is **damped free vibration.** When frictional forces are also absent, the motion is **undamped free vibration.** If a disturbing force acts on a structure, the resulting motion is **forced vibration** (see also Art. 3.53).

In Art. 3.36, the concept of a degree of freedom is introduced. Similarly, in the context of dynamics, a structure will have n degrees of freedom if n displacement components are required to define the deformation of the structure at any time. For example, a mass M attached to a spring with a negligible mass compared with M represents a one-degree-of-freedom system (Fig. 99*a*). A two-mass system interconnected by weightless springs (Fig. 3.99*b*) represents a two-degree-of-freedom system. The beam with the uniformly distributed mass in Fig. 3.99*c* has an infinite number of degrees of freedom because an infinite number of displacement components are required to completely describe its deformation at any instant of time.

Because the behavior of a structure under dynamic loading is usually complex, corresponding analyses are generally performed on idealized representations of the structure. In such cases, it is often convenient to represent a structure by one or more dimensionless weights interconnected to each other and to fixed points by weightless springs. For example, the dynamic behavior of the beam shown in Fig. 3.99*c* may be approximated by lumping its distributed mass into several concentrated masses along the beam. These masses would then be joined by members that have bending stiffness but no mass. Such a representation is often called an **equivalent lumped-mass model.** Figure 3.99*d* shows an equivalent four-degree-of-freedom, lumped-mass model of the beam shown in Fig. 3.99*c* (see also Art. 3.53).

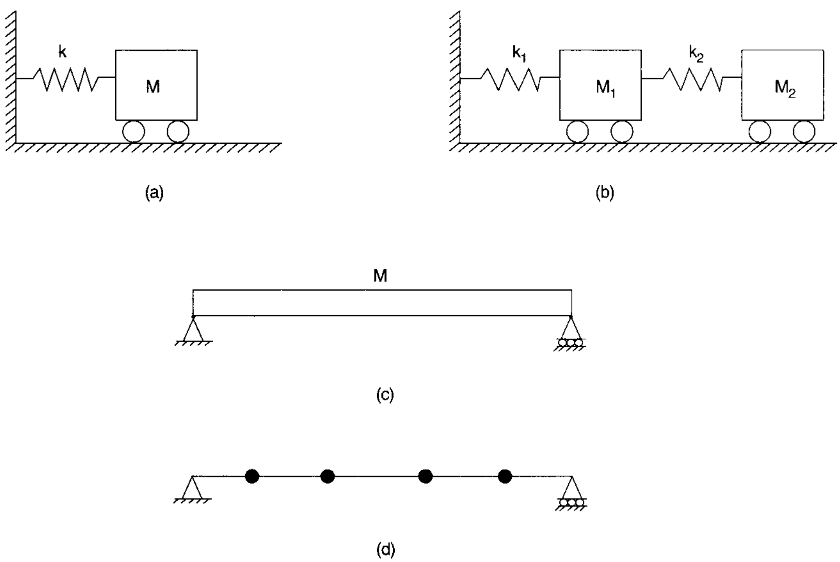


FIGURE 3.99 Idealization of dynamic systems. (*a*) Single-degree-of-freedom system. (*b*) Two-degree-of-freedom system. (*c*) Beam with uniformly distributed mass. (*d*) Equivalent lumped-mass system for beam in (*c*).

3.53 VIBRATION OF SINGLE-DEGREE-OF-FREEDOM SYSTEMS

Several dynamic characteristics of a structure can be illustrated by studying single-degree-of-freedom systems. Such a system may represent the motion of a beam with a weight at center span and subjected to a time-dependent concentrated load $P(t)$ (Fig. 3.100*a*). It also may approximate the lateral response of a vertically loaded portal frame constructed of flexible columns, fully restrained connections, and a rigid beam that is also subjected to a time-dependent force $P(t)$ (Fig. 3.100*b*).

In either case, the system may be modeled by a single mass that is connected to a weightless spring and subjected to time-dependent or dynamic force $P(t)$ (Fig. 3.100*c*). The magnitude of the mass m is equal to the given weight W divided by the acceleration of gravity $g = 386.4$ in/sec^2. For this model, the weight of structural members is assumed negligible compared with the load W. By definition, the stiffness k of the spring is equal to the force required to produce a unit deflection of the mass. For the beam, a load of $48EI/L^3$ is required at center span to produce a vertical unit deflection; thus $k = 48EI/L^3$, where E is the modulus of elasticity, psi; I the moment of inertia, in^4; and L the span of the beam, in. For the frame, a load of $2 \times 12EI/h^3$ produces a horizontal unit deflection; thus $k = 24EI/h^3$, where I is the moment of inertia of each column, in^4, and h is the column height,

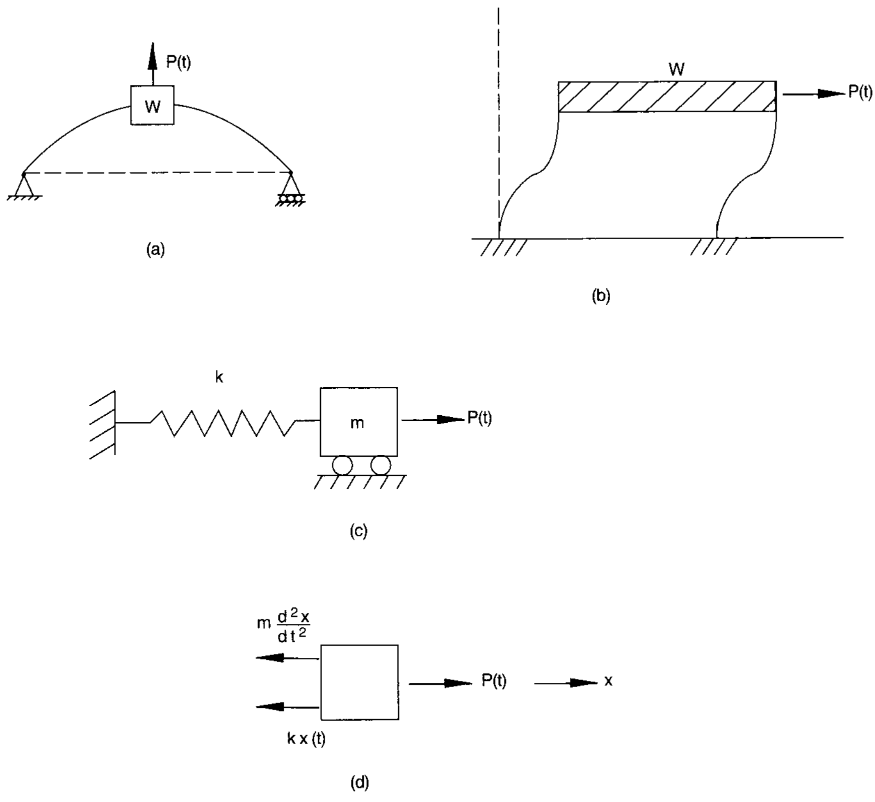


FIGURE 3.100 Dynamic response of single-degree-of-freedom systems. Beam (*a*) and rigid frame (*b*) are represented by a mass on a weightless spring (*c*). Motion of mass (*d*) under variable force is resisted by the spring and inertia of the mass.

in. In both cases, the system is presumed to be loaded within the elastic range. **Deflections are assumed to be relatively small.**

At any instant of time, the dynamic force $P(t)$ is resisted by both the spring force and the inertia force resisting acceleration of the mass (Fig. 3.100*d*). Hence, by d'Alembert's principle (Art. 3.7), dynamic equilibrium of the body requires

$$m \frac{d^2x}{dt^2} + kx(t) = P(t) \tag{3.190}$$

Equation (3.190) represents the controlling differential equation for modeling the motion of an undamped forced vibration of a single-degree-of-freedom system.

If a dynamic force $P(t)$ is not applied and instead the mass is initially displaced a distance x from the static position and then released, the motion would represent undamped free vibration. Equation (3.190) reduces to

$$m \frac{d^2x}{dt^2} + kx(t) = 0 \tag{3.191}$$

This may be written in the more popular form

$$\frac{d^2x}{dt^2} + \omega^2(t) = 0 \tag{3.192}$$

where $\omega = \sqrt{k/m}$ = **natural circular frequency,** radians per sec. Solution of Eq. (3.192) yields

$$x(t) = A \cos \omega t + B \sin \omega t \tag{3.193}$$

where the constants A and B can be determined from the initial conditions of the system.

For example, if, before being released, the system is displaced x' and provided initial velocity v', the constants in Eq. (3.193) are found to be $A = x'$ and $B = v'/\omega$. Hence the equation of motion is

$$x(t) = x' \cos \omega t + v' \sin \omega t \tag{3.194}$$

This motion is periodic, or harmonic. It repeats itself whenever $\omega t = 2\pi$. The time interval or **natural period of vibration *T*** is given by

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \tag{3.195}$$

The **natural frequency** f, which is the number of cycles per unit time, or hertz (Hz), is defined as

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \tag{3.196}$$

For undamped free vibration, the natural frequency, period, and circular frequency depend only on the system stiffness and mass. They are independent of applied loads or other disturbances.

(J. M. Biggs, *Introduction to Structural Dynamics;* C. M. Harris and C. E. Crede, *Shock and Vibration Handbook,* 3rd ed.; L. Meirovitch, *Elements of Vibration Analysis,* McGraw-Hill, Inc., New York.)

3.54 MATERIAL EFFECTS OF DYNAMIC LOADS

Dynamic loading influences material properties as well as the behavior of structures. In dynamic tests on structural steels with different rates of strain, both yield stress and yield strain increase with an increase in strain rate. The increase in yield stress is significant for A36 steel in that the average dynamic yield stress reaches 41.6 ksi for a time range of loading between 0.01 and 0.1 sec. The strain at which strain hardening begins also increases, and in some cases the ultimate strength can increase slightly. In the elastic range, however, the modulus of elasticity typically remains constant. (See Art. 1.11.)

3.55 REPEATED LOADS

Some structures are subjected to repeated loads that vary in magnitude and direction. If the resulting stresses are sufficiently large and are repeated frequently, the members may fail because of fatigue at a stress smaller than the yield point of the material (Art. 3.8).

Test results on smooth, polished specimens of structural steel indicate that, with complete reversal, there is no strength reduction if the number of the repetitions of load is less than about 10,000 cycles. The strength, however, begins to decrease at 10,000 cycles and continues to decrease up to about 10 million cycles. Beyond this, strength remains constant. The stress at, this stage is called the **endurance,** or **fatigue, limit.** For steel subjected to bending with complete stress reversal, the endurance limit is on the order of 50% of the tensile strength. The endurance limit for direct stress is somewhat lower than for bending stress.

The fatigue strength of actual structural members is typically much lower than that of test specimens because of the influences of surface roughness, connection details, and attachments (see Arts. 1.13 and 6.22).